Comment on "Stokes' first problem for an Oldroyd-B fluid in a porous half space" [Phys. Fluids 17, 023101 (2005)]

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We point out and correct a significant error in the analytical solution presented by Tan and Masuoka [Phys. Fluids **17**, 023101 (2005)]. © *2009 American Institute of Physics.*
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In a 2005 paper, Tan and Masuoka[1] considered a generalization of Stokes' first problem that involves the flow of an Oldroyd-B fluid in a porous half space. These authors used the Fourier sine transform to obtain, what they believed to be, the exact solution of this initial-boundary value problem (IBVP). Unfortunately, due to the omission of a critical term from the resulting subsidiary equation, a mistake not uncommon in the literature, the solution given in Ref. 1 is, generally speaking, incorrect.

The aim of the present comment is to point out and correct this error, as well as a few misprints that also appear in Ref. 1. Hence, employing (with some additions) the same notation convention used in Ref. 1, we begin by restating the IBVP considered by Tan and Masuoka,

$$(1+\alpha\,\partial/\partial t)\,\partial u/\partial t=(1+\alpha_t\,\partial/\partial t)(\partial^2 u/\partial y^2-\beta^2 u),\qquad (y,t)\in(0,\infty)(0,\infty),\tag{1a}$$

$$u(0,t)=H(t),\quad u(\infty,t)=0,\quad t>0,\tag{1b}$$

$$u(y,0)=0,\quad \partial u(y,0)/\partial t=0,\quad y>0,\tag{1c}$$

where $H(\cdot)$ denotes the Heaviside unit step function and $\alpha(>0)$, $\alpha_t(\geq 0)$, and $\beta(\geq 0)$ are constants.

On applying the Fourier sine transform (see Sec. 138 of Ref. 2) with respect to y to Eq. (1a), and then employing the boundary conditions given in Eq. (1b), we obtain the subsidiary equation

$$\alpha\frac{d^2\bar{u}}{dt^2}+[1+\alpha_t(s^2+\beta^2)]\frac{d\bar{u}}{dt}+(s^2+\beta^2)\bar{u}=sH(t)+s\alpha_t\delta(t).\tag{2}$$

Here, $\bar{u}$ denotes the Fourier sine transform of u, with parameter s, and $\delta(\cdot)$ denotes the Dirac delta function. It is at this stage that the error in Ref. 1 occurred; specifically, the subsidiary equation, Eq. (23) of Ref. 1, does *not* contain the last term, which represents the contribution from $dH(t)/dt$, that appears on the right-hand side (RHS) of Eq. (2).

On solving Eq. (2) using the Laplace transform, and then making use of the initial conditions, Eq. (1c), we obtain the dual-transform domain solution

$$\hat{\bar{u}}(s,p)=\frac{s(1+\alpha_t p)}{p\{\alpha p^2+[1+\alpha_t(s^2+\beta^2)]p+s^2+\beta^2\}},\tag{3}$$

where $\hat{\bar{u}}$ denotes the Laplace transform of $\bar{u}$ and p is the Laplace transform parameter. Using partial fractions and a standard table of inverses (see, e.g., Ref. 2), the exact inverse Laplace transform of Eq. (3) can be readily determined. Omitting the details, it can be shown that $\bar{u}(s,t)=H(t)\times[s(s^2+\beta^2)^{-1}-\bar{U}(s,t)]$, where

$$\bar{U}(s,t)=s\begin{cases}\dfrac{\exp[-g(s)t]\{\sqrt{f(s)}\cosh[t\sqrt{f(s)}]+g(s)\sinh[t\sqrt{f(s)}]\}}{\sqrt{f(s)}(s^2+\beta^2)}-\dfrac{\alpha_t\exp[-g(s)t]\sinh[t\sqrt{f(s)}]}{\alpha\sqrt{f(s)}}, & f(s)>0,\\[2ex] \dfrac{\exp[-g(s)t]\{\sqrt{|f(s)|}\cos[t\sqrt{|f(s)|}]+g(s)\sin[t\sqrt{|f(s)|}]\}}{\sqrt{|f(s)|}(s^2+\beta^2)}-\dfrac{\alpha_t\exp[-g(s)t]\sin[t\sqrt{|f(s)|}]}{\alpha\sqrt{|f(s)|}}, & f(s)<0,\end{cases}\tag{4}$$

$$f(s)=\frac{\alpha_t^2(s^2+\beta^2)^2-2(2\alpha-\alpha_t)(s^2+\beta^2)+1}{4\alpha^2},\tag{5a}$$

$$g(s)=\frac{1+\alpha_t(s^2+\beta^2)}{2\alpha}\quad[g(s)>0],\tag{5b}$$

and we observe that the last term in each case of Eq. (4) represents the contribution from the last term on the RHS of Eq. (2).

On multiplying $\bar{u}(s,t)$ by $(2/\pi)\sin(sy)$ and then integrating with respect to s from zero to infinity, we obtain

$$u(y,t)=H(t)\left[e^{-\beta y}-\frac{2}{\pi}\int_0^\infty\bar{U}(s,t)\sin(sy)\,ds\right],\tag{6}$$

which is the exact yt-domain solution for $\alpha>0$. (For a treatment of the limiting case $\alpha\to 0$, see Ref. 3.) Of course, one must be mindful of the fact that it is possible for f to change sign as $s\to\infty$ from zero. In practical terms, this means determining the integration breakpoints, i.e., the positive roots (if

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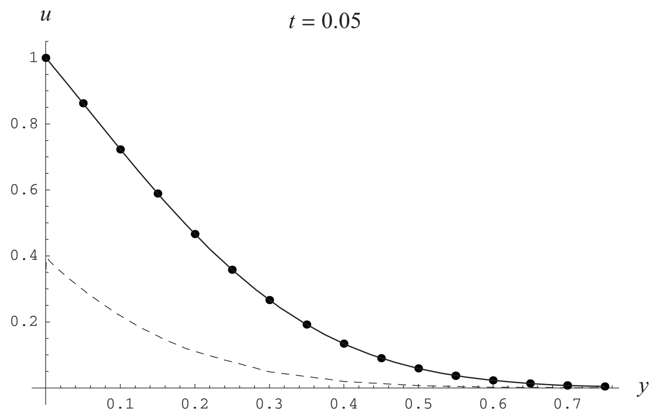

FIG. 1. u vs y for $\beta=0$, $\alpha=0.15$, and $\alpha_t=0.1$. Broken: Eq. (32) of Ref. 1. Dots: Eq. (7). Solid: Numerically generated inverse of Eq. (9) using Eq. (10) with $M=10\,000$.

any) of the equation $f(s)=0$ for all possible values of the parameters. Unfortunately, due to the lack of space, we are only able to give details on two of the many special cases.

Assuming first that $\alpha_t>0$ and $\beta=0$, Eq. (6) becomes

$$u(y,t)=H(t)\begin{cases} 1-\dfrac{2}{\pi}\Bigg[\displaystyle\int_0^{s_1^\bullet}\bar{\mathcal{U}}_+(s,t)\sin(sy)ds \\ \quad +\displaystyle\int_{s_1^\bullet}^{s_2^\bullet}\bar{\mathcal{U}}_-(s,t)\sin(sy)ds \\ \quad +\displaystyle\int_{s_2^\bullet}^{\infty}\bar{\mathcal{U}}_+(s,t)\sin(sy)ds\Bigg], \quad \alpha>\alpha_t, \\ \operatorname{erfc}\left(\dfrac{1}{2}yt^{-1/2}\right), \quad \alpha=\alpha_t, \\ 1-\dfrac{2}{\pi}\displaystyle\int_0^{\infty}\bar{\mathcal{U}}_+(s,t)\sin(sy)ds, \quad \alpha<\alpha_t, \end{cases} \tag{7}$$

where erfc$(\cdot)$ denotes the complementary error function, $\bar{\mathcal{U}}_{\pm}(s,t)=\bar{U}(s,t)|_{\beta=0}$ for $f(s)\gtrless 0$, and

$$s_{1,2}^\bullet=\alpha_t^{-1}\sqrt{2\alpha-\alpha_t\mp 2\sqrt{\alpha(\alpha-\alpha_t)}} \quad (\alpha_t>0). \tag{8}$$

To provide a partial, but independent, check of our work, we have in Fig. 1 plotted the solution profiles corresponding to Eq. (32) of Ref. 1, Eq. (7), and (see Ref. 4)

$$\hat{u}(y,p)=\frac{1}{p}\exp[-y\sqrt{(1+\alpha p)/(\alpha_t+1/p)}] \quad (\beta=0), \tag{9}$$

the latter being the solution of IBVP (1) in the Laplace transform domain, for the case $\alpha>\alpha_t$. Here, Eq. (9) was inverted numerically using Tzou's[5] formula, i.e.,

$$\mathfrak{F}(y,t)\approx t^{-1}e^{4.7}\left\{\frac{1}{2}\hat{\mathfrak{F}}(y,4.7/t)+\operatorname{Re}\left[\sum_{m=1}^{M}(-1)^m \times\hat{\mathfrak{F}}(y,(4.7+im\pi)/t)\right]\right\} \quad (t>0), \tag{10}$$

where $M(\geqslant 1)$ is an integer and Re$(\cdot)$ denotes the real part of a complex quantity. Clearly, the profiles corresponding to Eqs. (7) and (9) are in excellent agreement.

Tan and Masuoka also included results for the special case $\alpha_t=0$. While these expressions are not impacted by the aforementioned error, a close examination reveals that two of them, Eqs. (33)–(34) and Eq. (35) of Ref. 1, either contain a misprint or are not fully reduced. Thus, noting that the latter is the $\beta=0$ special case of the second, we point out, for completeness, that Eqs. (33) and (34) of Ref. 1 should read

$$u(y,t)=H(t)\times\begin{cases} e^{-\beta y}-2\pi^{-1}e^{-\sigma t}\Bigg[\displaystyle\int_0^{s^*}\bar{\mathfrak{U}}_+(s,t)\sin(sy)ds \\ \quad +\displaystyle\int_{s^*}^{\infty}\bar{\mathfrak{U}}_-(s,t)\sin(sy)ds\Bigg], \quad \beta<\dfrac{1}{2\sqrt{\alpha}}, \\ \exp\left(-y\sqrt{\dfrac{1}{2}\sigma}\right)H(t-y\sqrt{\alpha}), \quad \beta=\dfrac{1}{2\sqrt{\alpha}}, \\ e^{-\beta y}-2\pi^{-1}e^{-\sigma t}\displaystyle\int_0^{\infty}\bar{\mathfrak{U}}_-(s,t)\sin(sy)ds, \quad \beta>\dfrac{1}{2\sqrt{\alpha}}, \end{cases} \tag{11}$$

which we obtained from Eq. (6) by setting $\alpha_t=0$. In Eq. (11), $\bar{\mathfrak{U}}_{\pm}(s,t)=\bar{U}(s,t)|_{\alpha_t=0}$ for $f(s)\gtrless 0$ and we have set $\sigma=1/(2\alpha)$ and $s^*=\sqrt{(4\alpha)^{-1}-\beta^2}$ for convenience. Also, we observe that the quantity a (see the fourth page of Ref. 1) reduces to s^* when $\alpha_t=0$ and that Eq. (11) is, in fact, a solution of the well known telegraph equation.

Finally, as alluded to above, it is regrettable that, in addition to Ref. 1, the error described/corrected here has occurred in at least three other recent works, namely, Refs. 6–8.

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