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REPELLING SOLITON COLLISIONS IN COUPLED SCHRÖDINGER EQUATIONS WITH NEGATIVE CROSS MODULATION

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Abstract. The system of Coupled Nonlinear Schrödinger's Equations (CNLSE) is solved numerically by means of a conservative difference scheme. A new kind of repelling collision is discovered for negative values of the cross-modulation coupling parameter, α_2. The results show that as the latter becomes increasingly negative, the behavior of the solitons during interaction change drastically. While for $\alpha_2 > 0$, the solitons pass through each other, a negative threshold value $\alpha_2^* < 0$ is found below which the solitons repell each other. This is a novel result for this kind of models and the conservation of momentum for the system of quasi-particles (QPs) is thoroughly investigated.

1. **Introduction.** Numerous phenomena of physical significance are described by the various forms of the Nonlinear Schrödinger Equations (NLS). Topics in nonlinear optics, quantum fluids/condensed matter physics, gravitation, biological modelling, plasma physics, and many others are modelled with members of this class of equations. The single NLS has the form

$$i\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0. \tag{1}$$

As far as the applications in nonlinear optics are concerned, the above equation describes the single-mode wave propagation in a fiber. Depending on the sign of α, eq.(1) admits single and multiple sech-solutions ("bright solitons"), as well as tanh-profile, or "dark soliton" solutions. In this paper we concentrate on the case of bright solitons. In many cases, the fibers also allow propagation of more than one orthogonally polarized propagating modes, which may be described by a multicomponent version of (1) [17, 18].

When the physical system considered includes a coupling then a multicomponent version of the NLSE is to be used. The model involving Coupled Nonlinear Schrödinger Equations (CNLSE), is used to uncover a wealth of information about a wide variety of phenomena: interaction between pulses in nonlinear optics; Bose-Einstein condensates; signals in nonlinear acoustic media, etc. The generic CLNSE read:

$$\begin{aligned} \mathrm{i}\psi_t + \beta\psi_{xx} + [\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2]\Psi + \gamma\psi + \Gamma\phi = 0 \\ \mathrm{i}\phi_t + \beta\phi_{xx} + [\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2]\Phi + \gamma\phi + \Gamma\psi = 0 \end{aligned} \tag{2}$$

The parameter of interest for this work is α_2, or the cross-phase modulation, which defines the integrability of Eqs.(2). The Γ-term is responsible for strong coupling

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of the equations and has been treated in a previous work by the present authors. It is also referred to as linear birefringence [6], or relative propagation constant [26]. The term proportional to α_1, describes the self-focusing of a signal for pulses in birefringent media [18]. The parameter β describes the group velocity dispersion. Finally, the term γ appears as constant ambient potential, called normalized birefringence [5]. In this paper we are not concerned with the linear coupling and set $\gamma = 0$ and $\Gamma = 0$. More on the influence of Γ can be found in a previous paper of the present authors [22].

In terms of a general Schrödinger equation, we may view these various terms as potentials. Specifically, the term proportional to $\hat{\alpha} = \alpha_1 + 2\alpha_2$ can be seen as an interaction potential between evolving pulses in the CNLSE. In this work we intend to vary this interaction potential to uncover a new collision behavior, namely repulsion.

For $\Gamma = 0$, Eq.(2) is alternately called the Gross-Pitaevskii equation or an equation of Manakov type. It was derived independently by Gross ([14, 13] and Pitaevskii [20], to describe the behavior of Bose-Einstein condensates as well as optic pulse propagation. It was solved analytically for the case $\alpha_2 = 0, \beta = \frac{1}{2}$ by Manakov [17] via Inverse Scattering Transform who generalized an earlier result by Zakharov and Shabat [27, 28] for the scalar cubic NLSE (i.e. Eq.(2-ψ) with $\phi(x,t) = 0$). Recently, Chow, Nakkeeran, and Malomed [7] studied periodic waves in optic fibers using a version of (2) with $\Gamma \neq 0$.

In this paper we focus on the collision properties of solitons that initially have linear polarization, namely

$$\begin{pmatrix}\psi_l(x,t)\\ \phi_l(x,t)\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix}\Omega(x,t;c,n,X)\\ 0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\psi_r(x,t)\\ \phi_r(x,t)\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix}0\\ \Omega(x,t;c,n,X)\end{pmatrix},$$

where the subscripts l and r denote left or right initial quasi-particle (QP). Respectively, Ω is the one-soliton solution of the NLSE, which is given by

$$\Omega(x,t;c,n,X) = A\text{Sech}[b(x - X - ct)]\exp[\imath\frac{c}{2}x + n), \tag{3}$$

where c is the phase speed and n is the carrier frequency. The one-soliton solution Eq. (3) exist, when its amplitude is related to c and n as follows

$$A = \sqrt{\frac{2\beta}{\alpha_1}}b, \quad b = \sqrt{n + \frac{c^2}{4}}. \tag{4}$$

The amplitude, A, depends on the envelope phase speed, c and on the carrier frequency, n. These *sech*-profile solutions, of special interest to optics, are referred to as bright solitons. The parameter α_1 describes a self-interaction of the signal. For example, in the study of Bose Einstein condensates, α_1 is the measure of interparticle interaction of the population in question. For pulses in birefringent media [18], it describes the self-focusing of a signal. The solitons suffer no deformation under collision but exhibit a phase shift, resulting in a change in the relative separation.

The functions ψ and ϕ have various interpretations in the context of optic pulses. These include the amplitudes of x and y polarizations in a birefringent nonlinear planar waveguide [5, 19] and the amplitudes of x and y linear polarizations in a birefringent fiber [5, 4]; pulsed wave amplitudes of left and right circular polarizations [6, 3, 26] and pulsed wave amplitudes of symmetric and antisymmetric modes in a nonlinear coupler [21, 15].

For some regimes of nonzero α_2, the solitons experience deformation after a collision manifested in the creation of "shadow pulses" since these deformations often take the form of stable oscillations tied to the soliton in question. The oscillations are left behind the initial pulse, and they are also called "tail modes." In keeping with [9, 22] we will refer to these as inelastic collisions because they give rise to excitation of tail modes.

To the limit of our knowledge, the case $\alpha_2 < 0$ has not been treated in detail in the literature. In the present paper we set the goal to unearth the effect of negative cross-modulations on the properties of the soliton collisions in NLSE. To this end, we focus our attention on interacting bright solitons for which the asymptotic boundary conditions (a.b.c.) read

$$\psi, \psi_x, \phi, \phi_x \to \ 0, \qquad \text{for} \qquad |x| \to \ \ \infty \tag{5}$$

with initial conditions $\psi(x,0) = \psi_o(x)$ and $\phi(0,t) = \phi_o(x)$ that are consistent with the a.b.c.

2. **Conservation Laws.** CNLSE, Eqs.(2), possess two standard conserved quantities: mass and energy. We can obtain the conservation law for the wave mass by multiplying the first and second parts of (2) by $\overline{\psi}$ and $\overline{\phi}$, respectively, collecting the imaginary terms, and then integrating over space. This yields

$$\frac{\partial}{\partial t}\boldsymbol{M} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\left(|\psi|^2 + |\phi|^2\right)dx = 0 \tag{6}$$

Notice that the "masses" of the functions ψ and ϕ are not conserved separately. This differs from the case $\Gamma = 0$, when separate conservation laws are found, namely

$$\frac{\partial}{\partial t}\boldsymbol{M}_\psi = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\psi|^2 dx = 0, \quad \frac{\partial}{\partial t}\boldsymbol{M}_\phi = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\phi|^2 dx = 0 \tag{7}$$

The conservation of energy can be derived by multiplying the first and second parts of Eqs. (2) by $\overline{\psi}_t$ and $\overline{\phi}_t$, respectively, collecting the real terms, and integrating over space. This yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E} = \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} & -\beta[|\psi_x|^2 + |\phi_x|^2] + \frac{\alpha_1}{2}[|\psi|^4 + |\phi|^4] \\ & + (\alpha_1 + 2\alpha_2)[|\psi|^2|\phi|^2] + \gamma[|\psi|^2 + |\psi|^2] + 2\Gamma Re[\phi^*\psi]dx = 0\end{aligned} \tag{8}$$

Similarly to the case with the "wave mass", M, the energy is not conserved individually for ψ and ϕ which is a manifestation of the role of coupling in Eqs (2). We do not dwell on the problems connected with the linear coupling, and in our case, the masses connected with each component of the solution are conserved separately.

3. **Numerical Scheme.** Since the system we study is non-integrable, a numerical approach must be used to find approximate solutions of Eqs. (2). In the last four decades, different numerical schemes have been developed in the literature for generalized wave equations containing dispersion (DGWE) and for nonlinear evolution equations (NEE) that result from the DGWE in a moving frame. A main representative of the NEE is the Kortewrg-de Vries Equation (KdV) and its different modifications. Originally, most of these equations appeared in the shallow-water theory, but significant attention was also paid to waves on layers of infinite depth. The other beneficiary of numerical methods for DGWE appears to be the nonlinear optics, where different versions of the Nonlinear Schrödinger Equation (NLSE) appear for modeling the propagation of beams in fiber optics.

Numerical works which influenced the present paper include the pioneering effort [12] where a highly accurate spectral scheme was developed (see also [11] for the KdV equation (shallow water)). For water of infinite depth, a numerical scheme was developed in [16]. A decisive advance was made in [1, 2] where the schemes were designed in a manner better suiting inverse scattering techniques. Numerical approaches were further advanced in [24] for KdV and [23] for the NLSE. The last paper sets the standard in NLSE computations, and is the most relevant to our approach. The scheme used in the present work reflects the system's dynamic properties: specifically, we construct a scheme which mimics the integral constants of the system. It is descendent of [9, 22], and differs from [24, 23] because it makes use of internal iterations, and because it is about the Coupled NLSE (also known as 'vector' NLSE). The latter does not permit us to perform a head-to-head comparison with the above mentioned schemes, but the performance of our scheme was validated by all standard tests in numerical analysis: doubling and halving the spacing and the time increment; increasing and decreasing the cut-of of the spatial interval emulating the infinite region, etc. Conservation of energy is especially important in the context of long-time calculations, where non-conservative methods can produce spurious data for the energy. The scheme of [9, 22] has recently been extended to a fully complex version [25]. For the sake of self-consistency of the paper, we briefly summarize the numerical algorithm here.

Consider a uniform mesh in the interval [-L_1, L_2], namely $x_i = (i-1)h$, where $h = (L_1 + L_2)/(N-1)$ and N is the total number of grid points in the interval. Let τ be the time discretization. Respectively, ψ_i^n and ϕ_i^n denote the values of ψ and ϕ at the i^{th} spatial point and at the time nτ. Consider the scheme

$$\begin{aligned} \mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} &= \frac{\beta}{2h^2}[\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^{n} - 2\psi_i^{n} + \psi_{i+1}^{n}] \\ &\quad + \frac{\psi_i^{n+1,k}+\psi_i^n}{4}\big\{\alpha_1[|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2] \\ &\qquad + (\alpha_1+2\alpha_2)[|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2]\big\} \end{aligned} \tag{9a}$$

$$\begin{aligned} \mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} &= \frac{\beta}{2h^2}[\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^{n} - 2\phi_i^{n} + \phi_{i+1}^{n}] \\ &\quad + \frac{\phi_i^{n+1,k}+\phi_i^n}{4}\big\{\alpha_1[|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2] \\ &\qquad + (\alpha_1+2\alpha_2)[|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2]\big\}. \end{aligned} \tag{9b}$$

We conduct the internal iterations (repeating time steps) until convergence, i.e. $\|\psi^{n+1,k+1} - \psi^{n+1,k+1}\|/\|\psi^{n+1,k+1}\| \leq 10^{-12}$. Suppose that the internal iterations converge. Then at each time step we have the solution of the following nonlinear scheme

$$\begin{aligned} i\frac{\psi_i^{n+1}-\psi_i^n}{\tau} &= \frac{\beta}{2h^2}[\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^{n} - 2\psi_i^{n} + \psi_{i+1}^{n}] \\ &\quad + \frac{\psi_i^{n+1}+\psi_i^n}{4}\big\{\alpha_1[|\psi_i^{n+1}|^2 + |\psi_i^n|^2] + (\alpha_1+2\alpha_2)[|\phi_i^{n+1}|^2 + |\phi_i^n|^2]\big\}, \end{aligned} \tag{10a}$$

$$i\frac{\phi_i^{n+1}-\phi_i^n}{\tau} = \frac{\beta}{2h^2}[\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n] + \frac{\phi_i^{n+1}+\phi_i^n}{4}\big\{\alpha_1[|\phi_i^{n+1}|^2 + |\phi_i^n|^2] + (\alpha_1+2\alpha_2)[|\psi_i^{n+1}|^2 + |\psi_i^n|^2]\big\}. \quad (10b)$$

Following [9, 22] (see also [8] for similar derivations for the Boussinesq Equation) we see that the scheme (10a) mass and energy, i.e. there exist discrete analogs of (6) and (8).

For all $n \geq 0$,

$$\boldsymbol{M}^{\,n} = \sum_i |\psi_i^n|^2 + |\phi_i^n|^2 = const, \quad (11a)$$

$$\boldsymbol{E}^{\,n} = \sum_i \frac{-\beta}{2h^2}[|\psi_{i+1}^{\,n} - \psi_i^{\,n}|^2 + |\phi_{i+1}^{\,n} - \phi_i^{\,n}|^2] + \frac{\alpha_1}{4}[|\psi_i^n|^4 + |\phi_i^n|^4] + \frac{\alpha_2}{2}[|\psi_i^n|^2|\psi_i^n|^2] + \frac{\gamma_r}{2}[|\phi_i^{\,n}|^2 + |\psi_i^{\,n}|^2] + \Gamma Re[\overline{\phi}_i^{\,n}\psi_i^n] = const. \quad (11b)$$

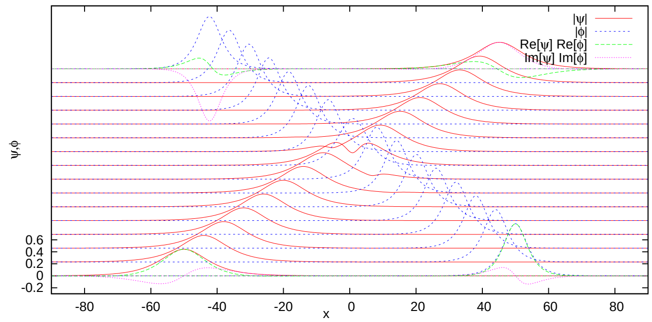
These values have been monitored during the computations, and we have fond that they are conserved within the round-off error, i.e., $M^{n+1} = M^n$ and $E^{n+1} = E^N$.

4. **Results and Discussion.** We call elastic a collision after which the original shapes re-emerge nondeformed while the inelastic is a collision in which additional oscillations are excited in the place of the bygone collision. The predominant part of the research in the literature deals with the positive values of the cross modulation, when the solitons pass through each other during the collisions. In optical applications, this kind of behavior is important and it seems reasonable to limit the investigation to positive cross-modulations. When investigating the quasi-particle behavior of the CNLSE solitons, one must focus on their interactions. In order to understand the analogy to the real particles, one needs to find a model that predicts repelling of the solitons after a collision. We have discovered that the solitons of the CNLSE with negative cross-modulation parameter can be repelled during collision. In this section we present our results on this new phenomenon.

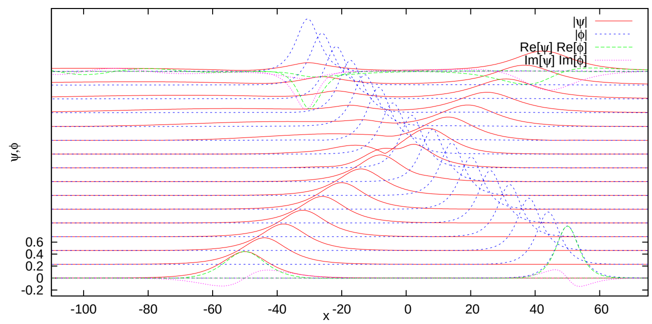
It should be noted that we have unearthed repelling interaction for different sets of values of the parameters, but will present here one case, in order not to overload the presentation with details. As already mentioned, we do not consider the linear copling, so we set $\gamma = 0$, $\Gamma = 0$. We also fix for simplicity the diffusion coefficient to $\beta = 1$, and the self-focusing to $\alpha_1 = 0.25$. For the bulk of numerical experiments, we set the two phase speeds of the left and tight solitons to $c_l = 0.15$, and $c_r = -0.15$, respectively. However, in order to have distinguishable Quasi-Particles (QPs), we chose different carrier frequencies for them, namely $n_l = 0.03$ and $n_r = 0.1$ for the above set of phase speeds.

Note that we have selected reasonably dense grid $h = 0.5$, and time step $\tau = 1$. The time step can be this large, because of the moderate values of the coefficients of nonlinearity $\alpha_{1,2}$ which are considered here. In terms of the dimensionless time, the processes are rather slow. Naturally, we have run cases with two and four times smaller τ and obtained results within a couple of percentages from the presented here.

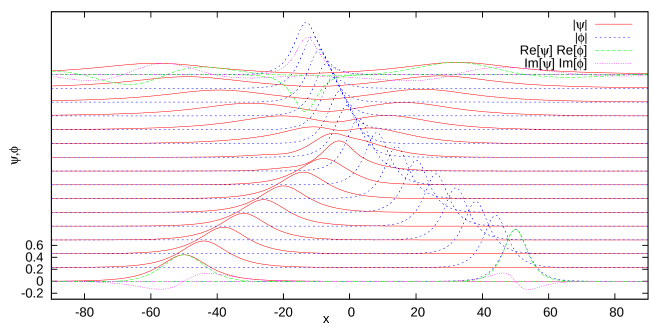
As already above mentioned, an analytical solution [17] is available for $\alpha_2 = 0$. Without being formally decoupled for $\alpha_2 = 0$, the CNLSE system behaves similarly to a decoupled system. As shown in Fig. 1-(a), our computation confirmed this fact. The other case when the system is decoupled is for $\alpha_2 = -\frac{\alpha_1}{2} = -0.125$ and we did verify that there is no whosoever interaction between the two QPs as the system is completely decoupled (not even a phase shift).



a) $\alpha_2 = 0, \alpha^* = 0.25$



b) $\alpha_2 = -0.1, \alpha^* = 0.05$



c) $\alpha_2 = -0.144, \alpha^* = -0.138$

FIGURE 1. Crossing collisions for $c_l = 1.$, $n_l = 0.03$, $c_r = 0.5$, $n_r = 0.1$

In order to understand better the interaction, we present the spatial profiles of the solution at different times as 'frozen movies.' The profiles for different moments of time are superimposed on the graph forming some kind of 'waterfall pattern.' On such a graph, it is easy to see the details of the interaction: both for the profiles and for the trajectories of the centers. The latter can be seen as the crests on the plot. In red color (solid line) is shown the modulo of component ψ, while $|\phi|$ is presented in blue (dashed line). Since the initial polarizations are linear, we have only ψ component for the left soliton, and only ϕ-component for the soliton on the right. In case that one of the components should excite the other one during the interactions (see, e.g. [25]), after the collisions the two solitons would exhibit both red- and blue-color lines. In the first and last moments of time under consideration, the real and imaginary parts of ψ, ϕ are presented in additional colors (or different length of dashes and dots).

Now, if α_2 becomes less negative than $-\frac{1}{2}$, and eventually even positive, we will be in the reasonably well investigated realm of crossing interactions of the solitons. For reference, we present in Fig. 1-(a) the case $\alpha_2 = 0$ for two solitons whose polarizations differ by 90°. Unlike the case $\alpha_2 = -\frac{1}{2}$ there is interaction between the QPs, but it is exactly as expected for an integrable case: the only manifestation of nonlinearity is the phase shift.

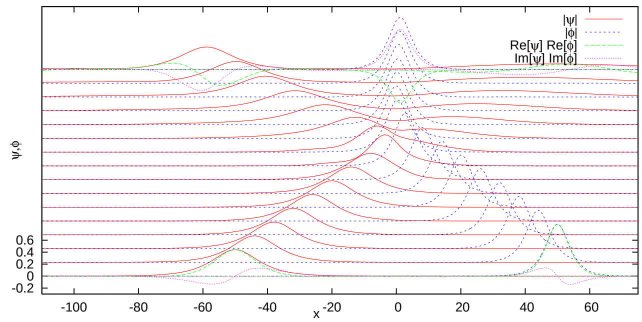
The absence of a cross-modulation manifests itself through the fact that the orthogonal components are not excited during the interaction. It was shown, e.g. in [25] for $\alpha_2 > 0$, that the orthogonal components are indeed excited after the interaction of two initially linearly polarized pulses. It is interesting to observe here in Fig. 1-(b) that in our parameter range, taking $\alpha_2 = -0.1$ does lead to some excitation of an orthogonal component, but only for the taller (and slower moving) QP. In a sense the faster QP was split into a reflected and refracted part, and the reflected part of it moves together with the other QP. It will take a special investigation to understand the role of the negative cross modulation when $\alpha_2 \in [-\frac{1}{2}\alpha_1, 0]$, because there are no cross-excitations that take place for the two boundary values in this interval, and the excitation for the rest of the values from that interval, are not similar to the case of positive cross-modulation.

The purpose of the present paper is to investigate the regimes when $\alpha_2 < -\frac{1}{2}\alpha_1$ or $\hat{\alpha} = \alpha_1 + 2\alpha_2 < 0$. We discovered that decreasing α_2 down from $-\frac{1}{2}\alpha_1$ leads to the increase of the amplitude of the reflected part of the faster QP and decrease of the refracted part. In addition, the excited orthogonal mode does not stick to the slower QP, but rather speeds away from it. In Fig. 1-(c) is shown the case $\alpha_2 = -0.144$ when the reflected and the refracted part have approximately equal amplitudes. At $\alpha_2 = -0.157$ all traces of the refracted part disappear.

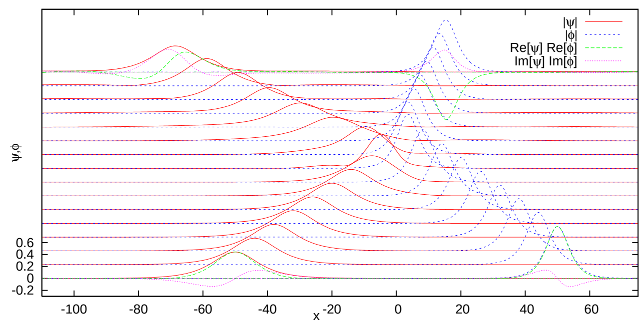
Actually, for the values of the parameters, considered here, the taller QP virtually stops and reverses its trajectory for $\alpha_2 = -0.15$ as shown in Fig 2-(a) For this value of α_2 there is still a small refracted part, but no excitations are observed.

Finally, in Fig 2-(b), we present a genuinely repelling interaction for $\alpha_2 = -0.2$. One can see that there is no excitation of the orthogonal components. The physical meaning of this observation is that a repulsive component appears in the potential of interaction of the QPs. The interaction potential becomes so negative that the QPs stop and turn back before they can get close enough to overlap. Thus the nonlinearity does not have the chance to excite the respective orthogonal mode. This is an important new result which is not very intuitive. Another model in which repulsion can take place is the sine-Gordon equation (sG), when the soliton-soliton interactions are investigated. The soliton-antisoliton interactions in sG are attractive. For the time being we have not implemented the coarse-grain approximation of [10] to the quasi-particles of this work, so we do not have an approximate analytical expression for the interaction potential, but the evidence unequivocally points out towards the presence of repulsive terms in the potential.

The most important observation here is that the trajectory of the smaller soliton (the one on the left has a smaller amplitude because of the smaller carrier frequency) bends more and it's final trajectory demonstrates that it is moving at higher phase speed than the one with which it enterred the collision. In this range of parameters,



(b) $\alpha_2 = -0.15, \alpha^* = -0.05$



(c) $\alpha_2 = -0.2, \alpha^* = -0.15$

FIGURE 2. Repelling interactions for negative values of the overall coupling parameter $\hat{\alpha} = \alpha_1 + 2\alpha_2$.

the trajectory of the larger quasi-particle appears as it is less bent and it returns with slower phase speed than its incident speed.

A most interesting result transpires from our results, namely that in the repelling collisions, there are no cross-excitations of the two components (modes) which appears to be a novel result shedding light on the soliton interactions for negative cross-modulations.

In order to confirm that the above reported behavior is authentic of the system, we investigated also a case with different initial carrier frequencies and different phase speeds. We took somewhat larger carrier frequencies which made left particle larger. For the specific numbers chosen, the heights of the two QPs are closer to each other. It is shown in Fig. 3 that once again, the qualitative nature of the interaction is the same: the QPs reflect completely without trace of refracted parts, or of excited signals. Because the larger QP is now not that much taller than the smaller one, the former acquires larger phase speed after the reflection. Respectively, the smaller QP is now slightly slower after the interaction. These are the quantitative difference, but the main qualitative feature is the same: a repelling interaction takes place for a negative α_2.

5. **Quasi-Particle (QP) Characteristics.** In the precedence, we have discovered strictly repelling QP collisions in CNLSE for negative cross-modulations. Upon collision not merely the sense of motion changes, but the magnitudes of the phase speeds assume different values, too. In the crossing interactions, the solitons return

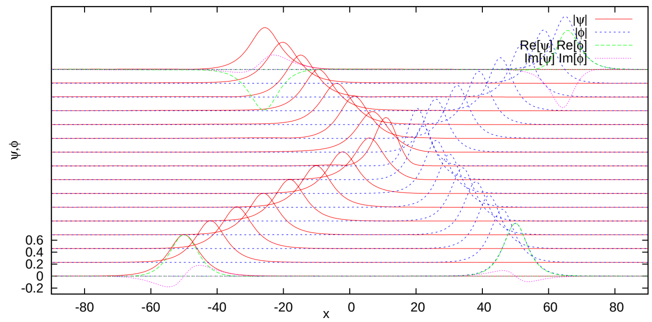


FIGURE 3. Repelling interactions for $\alpha_2 = -0.4$ and $c_l = 0.2$, $n_l = 0.07$, $c_r = 0.1$, $n_r = 0.1$

to their original phase speeds. Since it is not obvious how to quantify the phase shifts in this case, we will not dwell on this point here. Note here that due to the conservative nature of the numerical scheme, the total mass, energy, and pseudo-momentum of the system are conserved, despite the exotic nature of the collisions. In order to elucidate the quasi-particle aspects of the soliton collision in CNLSE, we computed the masses of the two QPs, according to Eq. (11a). We computed the phase speeds from the trajectories of the centers of QPs after fitting lines to them in the regions far ahead and far after the collision.

Computing the masses and phase speeds of the two QPs before and after the collision enables us to identify the particle-like properties of the collisions. We present the thus computed properties in Table 1. The upper part of the table deals with the initial values (superscript i), while the lower portion refers to the final values (superscript f). Letter M stands for mass, c_l, c_r stand for the phase speeds of the left and right QP.

The novel result of the present work is that for negative enough cross-modulations, a repelling type of interaction takes place, making the analogy between the solitons (quasi particles) and the real particles much more conspicuous. Then, it is only natural to examine the pseudo-Newtonian behavior of the QPs, e.g. to compute their celerities, and pseudomomenta before and after the collision. In the case of repelling collisions we are aided by the fact that there are no cross-excitations of the components.

Table 1 shows the main characteristics of the QPs before and after the collision, One sees that the individual masses and phase speeds change during the interaction which leads to different pseudomomenta. Yet the total pseudomomentum of the system of two QPs is conserved with very high degree of accuracy. In this sense, one is faced with Newton-like law for conservation of momentum of system of two quasi-particles.

6. **Conclusions.** In the present paper we consider a system of two Schrödinger equations coupled only through the nonlinear terms. We make use of a previously developed implicit conservative difference scheme. We fix all other parameters of the system, and focus on the case when the coupling parameter α_2 (called cross-modulation parameter) can become negative, and more specifically, when $\alpha_1 + 2\alpha_2 < 0$, where α_1 is the self-focusing parsmeter.

TABLE 1. Quasi-Particle characteristics

Before Collision							
α_2	M_l	M_r	c_l^i	c_r^i	$P_l^i = M_l c_l^i$	$P_r^i = M_r c_r^i$	$P_l^i + P_r^i$
-.2	0.2498	0.4915	0.1498	-0.1495	0.0374	-0.0735	-0.0361
-.25	0.2498	0.4915	0.1498	-0.1495	0.0374	-0.0735	-0.0361
-.3	0.2498	0.4915	0.1498	-0.1495	0.0374	-0.0735	-0.0361
-.4	0.2498	0.4915	0.1498	-0.1495	0.0374	-0.0735	-0.0361
-.4†	3.919	4.996	0.1994	-0.0997	0.7814	-0.4981	.2833
After Collision							
α_2	M_l	M_r	c_l^f	c_r^f	$P_l^f = M_l c_l^f$	$P_r^f = M_r c_r^f$	$P_l^f + P_r^f$
-.2	0.2498	0.4915	-0.2425	0.0495	-0.0606	0.0243	-0.0362
-.25	0.2498	0.4915	-0.2427	0.0503	-0.0606	0.0247	-0.0359
-.3	0.2498	0.4915	-0.2429	0.0504	-0.0607	0.0248	-0.0359
-.4	0.2498	0.4915	-0.2433	0.0506	-0.0608	0.0249	-0.0359
-.4†	3.919	4.996	-0.1350	0.1628	-0.5291	0.8133	.2843
† Case from Fig. 3							

We obtained results for a range of values of α_2. The system is completely decoupled for $\alpha_2 = -\frac{1}{2}\alpha_1$, and behaves as a single equation for $\alpha_2 = 0$, and we recovered the expected behavior numerically. Decreasing α_2 from $-\frac{1}{2}\alpha_1$ to more negative values, brought into view a completely new kind of behavior of the solitons upon collision: they repel each other instead of passing through one another as it is the case for positive cross-modulations.

The soliton waveforms are seen to survive the collision undeformed despite the fact that the system cannot be proved to be fully integrable. For α_2 sufficiently negative, the collisions become fully repelling, while the total energy is strictly conserved. We compute the masses and the pseudomomenta of the QPs before and after the collision and show that the total mass and the total pseudomomentum of the system of the QPs are conserved upon collision. The repelling interactions found in this work resemble one-dimensional ballistic collisions and further works will detail the depth to which this analogy may be carried.

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