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On the Solution of the System of ODEs Governing the Polarized Stationary Solutions of CNLSE

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Consider the coupled system of Nonlinear Schrodinger Equations (CNLSE):

$$\begin{aligned} \mathrm{i}\psi_t =& \beta\triangle\psi + [\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2]\psi \\ \mathrm{i}\phi_t =& \beta\triangle\phi + [\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)\alpha_2|\psi|^2]\phi \end{aligned}$$

where β is the dispersion coefficient, α_1 describes the self-focusing of a signal for pulses in birefringent media, and α_2 governs the nonlinear coupling between the equations. For further convenience we consider the case $\beta = 1$. All other cases can be reduced to this one by means of simple scaling. CNLSE possess solutions that are localized envelopes (see, for example [3, 8]). It is easy to see that if one of the components, say ϕ, is trivially equal to zero, then the system reduces to single scalar NLSE for the other function, ψ. Such a solution of CNLSE is called 'linearly polarized'. The *sech*-solution of the single NLSE, say for ψ-profile, is given by

$$\psi(x,t) = A_\psi \mathrm{sech}[b_\psi(x - X - c_\psi t)] \exp\Big\{\mathrm{i}\Big[\frac{c_\psi}{2\beta}(x - X - c_\psi t) - n_\psi\Big]\Big\} \qquad (1)$$

$$b_\psi^2 = \frac{1}{\beta}\big(n_\psi + \frac{1}{4\beta}c_\psi^2\big); \quad A_\psi = b_\psi\sqrt{2\beta/\alpha_1}; \quad u_c = 2n_\psi\beta/c_\psi,$$

which means that for given phase speed, c_ψ, and carrier frequencies, n_ψ, the solution of the above type is fully specified. The localized solution is identified by the presence of a spatial point, $x = X - c_\psi t$ ('center') which moves with a given phase speed c_ψ. The same initial relationships hold for ϕ-profile when $\psi = 0$. The center is common for both components ψ and ϕ. The above type of analytical initial condition evidently requires the restriction $4\beta\, b_{\psi,\phi} > -c_{\psi,\phi}^2$. It should be noted here, that for the Manakov case [3] when $\alpha_2 = 0$, a solution of type of Eq. (1) can be found for circular polarization of kind $\psi(x,t) = \chi(x,t)\cos\delta,\ \phi(x,t) = \chi(x,t)\sin\delta$, by replacing α_1 by $2\alpha_1$.

The initial conditions play important role in investigation of soliton interaction described by CNLSE. It has been shown in different works of the present authors [1, 4, 6, 7] that linearly polarized initial conditions, after the interaction invariable lead to solitons for which both $\psi, \phi \neq 0$, and have in general elliptic polarizations. This means that a numerical solution for the general case is needed in order to provide initial conditions with elliptic polarization which is the goal of the present work. A numerical solution for elliptic polarized stationary propagating solitons had been provided in [5]. In order to be able to use a numerically computed initial condition, one must have a code that gives the solution. Here we propose a different numerical method to find the solution to be used in our future work on interaction of solitons. Simultaneously, we can verify both our solution and the solution of [5].

The elliptically polarized solution has the form

$$\psi(x,t),\phi(x,t) = A_{\psi,\phi}(x - X - c_{\psi,\phi}t)\exp\left\{\mathrm{i}\big[n_{\psi,\phi} - \frac{c_{\psi,\phi}}{2}(x - X - c_{\psi,\phi}t) + \delta_{\psi,\phi}\big]\right\} \quad (2)$$

where n_ψ, n_ϕ are the carrier frequencies and δ_ψ, δ_ϕ are the phases for the two components. Generally $n_\psi \neq n_\phi$, while $n_\psi = n_\phi$ only for circular polarization. The general polarization angle is defined as $\theta = \arctan(\max\{A_\phi\}/\max\{A_\psi\})$.
After some straightforward manipulations we get the following system of conjugated real ordinary differential equations for the real-valued amplitudes A_ψ, A_ϕ:

$$\vec{F}(\vec{A}) \equiv \begin{pmatrix} A_\psi'' + (n_\psi + \frac{1}{4}c_\psi^2)A_\psi + [\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2)A_\phi^2]A_\psi \\ A_\phi'' + (n_\phi + \frac{1}{4}c_\phi^2)A_\phi + [\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2)A_\psi^2]A_\phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (3)$$

where $\vec{A} = [A_\psi(x - X - c_\psi t), A_\phi(x - X - c_\phi t)]^T$ is the unknown vector function.
The above system has to be solved with the asymptotic boundary conditions

$$\lim_{x\to\pm\infty} A'_{\psi,\phi}(x - X - c_{\psi,\phi}t) = \lim_{x\to\pm\infty} A_{\psi,\phi}(x - X - c_{\psi,\phi}t) = 0. \quad (4)$$

Thus we are faced with solving the bifurcation boundary-value problem (BVP) (3), (4) for the possible nontrivial solution. Introducing a fictitious time $\tau \geq 0$ let assume that the relationship holds

$$\frac{\partial \vec{F}}{\partial \vec{A}}\vec{\alpha} + \frac{\partial \vec{F}}{\partial \vec{A}''}\vec{\alpha}'' + \vec{F} = \vec{0}, \quad \text{with} \quad \vec{\alpha} \equiv \frac{\partial \vec{A}}{\partial \tau}. \quad (5)$$

It is proved in [2] that the iteration

$$\vec{A}^{k+1} = \vec{A}^k + \Delta\tau_k \vec{\alpha}^k, \quad \Delta\tau_k = \frac{||\vec{F}^k||}{||\vec{F}^k|| + ||\vec{F}^{k+1}||}, \quad k = 0, 1, 2, ...$$

is quadratically convergent in the vicinity of the solution of (3),(4). In the last formula $||\cdot||$ stands for the L_2-norm. It is convenient to use cubic Hermitean splines on the uniform mesh

$$x_i = -x_\infty + ih, \qquad h = \frac{2x_\infty}{N}, \qquad i = 0, ..., N,$$

and the Gaussian points $1/2 \pm \sqrt{3}/6$ in $[0, 1]$ to be the the collocation nodes.
We use sech-like function as an initial approximation, i.e., $A^{(0)}_{\psi,\phi} \propto \mathrm{sech}(b_{\psi,\phi}x)$ and solve the above auxiliary BVP for given $n_{\psi,\phi}$, $c_{\psi,\phi}$, α_1 and α_2 with spatial step $h = 10^{-2}$ and "actual" infinity $x_\infty = 15 \div 20$. Taking seven to eight iterations proves to be sufficient to reach the approximate nontrivial solution with L_2-norm of the error lesser than 10^{-12}.
In Fig. 1 we present three cases of elliptic polarization when the carrier frequency of the ψ-component is fixed ($n_\psi = -0.5$) and the carrier frequency of the ϕ-components $n_\phi \in [-0.7, -0.39]$. We were able to generate solutions with all possible angles of

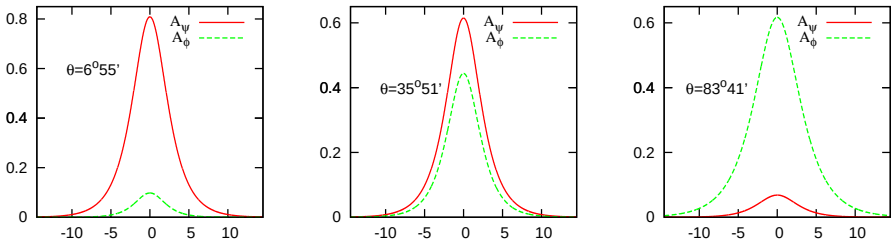

Figure 1: Amplitudes A_ψ and A_ϕ for $c_l = -c_r = 1$, α_1=0.75, $\alpha_2 = 0.2$, $n_\psi = -0.5$. Left: $n_\phi = -0.68$; middle: $n_\phi = -0.55$; right: $n_\phi = -0.395$.

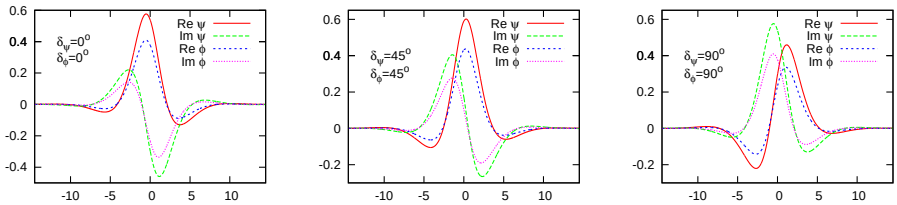

Figure 2: Real and imaginary parts of the amplitudes, $\Re\psi$, $\Im\psi$, $\Re\phi$, $\Im\phi$, from the case shown in the middle panel of Fig. 1 and the dependence on phase angle.

polarization in $\theta \in [0°, 90°]$ including the cases of circular and linear polarization. In this way we found out that the elliptically polarized solitons do exist even for those values of the carrier frequency and phase velocity, for which the linearly polarized sech-solutions do not exist. Note that the latter are limited by the condition $n_{\psi,\phi}+c^2_{\psi,\phi}/4 > 0$. Fig. 1 shows that by varying the carrier frequency one can produce solitons that have different supports. We mention here that as far as it can be judged by the graph, our shapes are in good quantitative agreement with the solution of [5].
Another dimension of complexity is introduced by the phases $\delta_{\psi,\phi}$ of the different components (see Eq. (2)) although these do not affect the amplitudes $A_{\psi,\phi}$. As shown in [7], the initial difference in phases can have a profound influence on the polarizations of the solitons after the interaction. In Fig. 2 we present a case with given amplitudes of envelops but with different phases. Naturally, we present both the real and imaginary parts, because their relative shift is what matters in this case. In the end, we present in Fig. 3 the interaction of two initial solitons with different polarizations. The left one has a circular polarization, while the right one has elliptic polarization. The above mentioned exchange of polarizations is clearly seen. The stability of the computations are ensured by the conservative scheme we employ, but the lack of oscillations and dispersion testifies that the initial conditions are with high accuracy compatible with the equations.

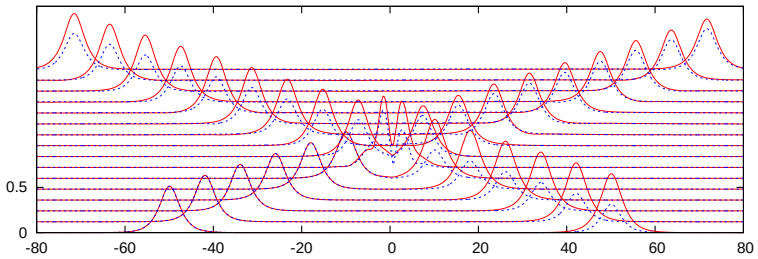

Figure 3: Collision dynamics of quasi-particles or $c_l = -c_r = 1$, α_1=0.75, $\alpha_2 = 0.2$. Left: circular polarization $n_{l\psi} = n_{l\phi} = -0.5$,. Right: elliptic polarization $n_{r\psi} = -0.55$, $n_{r\phi} = -0.46$.

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