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Collision dynamics of elliptically polarized solitons in Coupled Nonlinear Schrödinger Equations

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Abstract

We investigate numerically the collision dynamics of elliptically polarized solitons of the System of Coupled Nonlinear Schrödinger Equations (SCNLSE) for various different initial polarizations and phases. General initial elliptic polarizations (not *sech*-shape) include as particular cases the circular and linear polarizations. The elliptically polarized solitons are computed by a separate numerical algorithm. We find that, depending on the initial phases of the solitons, the polarizations of the system of solitons after the collision change, even for trivial cross-modulation. This sets the limits of practical validity of the celebrated Manakov solution. For general nontrivial cross-modulation, a jump in the polarization angles of the solitons takes place after the collision ('polarization shock'). We study in detail the effect of the initial phases of the solitons and uncover different scenarios of the quasi-particle behavior of the solution. In majority of cases the solitons survive the interaction preserving approximately their phase speeds and the main effect is the change of polarization. However, in some intervals for the initial phase difference, the interaction is ostensibly inelastic: either one of the solitons virtually disappears, or additional solitons are born after the interaction. This outlines the role of the phase, which has not been extensively investigated in the literature until now.

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Keywords: Coupled Nonlinear Schrödinger Equations; Elliptic polarization; Phase shift; Trapping effect

1. Introduction

The system of Coupled Nonlinear Schrödinger Equations (SCNLSE) is a soliton supporting system. It appeared initially as a model for light propagation in isotropic Kerr materials ([10,8,16,18]). Apart from its splendid performance in modeling the propagation of light pulses in fiber optics, it also offers the opportunity to investigate the quasi-particle behavior soliton. The latter is indispensable for the understanding of the fundamental phenomena associated with propagation of nonlinear waves. There are many different solitons supporting systems whose solution behave as quasi-particles, but NLSE and SCNLSE exhibit the richest behavior and serve as very important testing ground for the quasi-particle approach. For this reason, the SCNLSE attracted the attention of the leading researches, and a

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number of excellent analytical results have been obtained during the years. For the integrable Manakov case (see [14,9,1,2,12,15,17,3], among others), linear and circular polarizations were treated in [13], and general polarization in [26,11], among others.

The initial polarization of SCNSE and its evolution during the quasi-particle interaction is a very important element which is uniquely associated with the vector nature of the model. There are numerous experimental observations (see [7] and cited there literature) of the effects connected with the polarization. Another important application of the SCNLSE can have as a model case for testing the different approximate approaches to the quasi-particle facet of the solitons, such as variational approximation, coarse-grain description, etc. (see [6] and the literature cited therein).

The existence of analytical solutions is usually limited to simpler models, and for the full fledged SCNLS with nontrivial cross-modulation, no such solution is available. In addition, an analytical solution cannot answer the questions related to its physical significance before its stability or robustness (in some general sense) is established. This can be done, e.g., via adequately devised numerical scheme. For this reasons, the numerical schemes for SCNLS have a very important role to play. An implicit scheme of Crank–Nicolson type was first proposed for the single NSE as a tool for investigation in the extensive numerical treatise [21]. In order to adjust this scheme to SCNSLE, internal iterations were applied in [5]. This concept yielded both fully implicit scheme and implementation of the conservation laws on difference level within the round-off error of the calculations. The above scheme was extended to complex arithmetic in [22,23] where the computer code for Gaussian elimination with pivoting of [4] was generalized for complex-valued multi-diagonal band algebraic systems. As a result, the number of unknown functions was reduced in half and the computations became four times faster.

Having in mind the importance of initial polarization, we investigated in [24] the collision dynamics for circularly polarized solitons based on *sech*-functions and found out that there exist infinite number of Manakov two-soliton solutions preceded by polarization discontinuity (shock) on the place of interaction. An interesting new result was that Manakov solitons with 45° initial polarization can emerge unchanged from the collision even for nontrivial cross-modulation, provided that the initial phases of both QPs are equal to zero. In order to enrich the range of investigations in the case of general polarization we established an auxiliary conjugate system of nonlinear ordinary differential equations [25] in order to generate numerically initial elliptically polarized soliton solutions. In this work we aim to conduct series of simulations and to track the particle-like behavior of the interacted localized waves with arbitrary (elliptic) initial polarization.

2. Problem formulation

SCNLSE is system of nonlinearly coupled equations (called alternatively the Gross-Pitaevskii or Manakov-type system). In standard notations the system reads:

$$\mathrm{i}\psi_t = \beta\psi_{xx} + [\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2]\psi, \tag{1a}$$

$$\mathrm{i}\phi_t = \beta\phi_{xx} + [\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2]\phi, \tag{1b}$$

where β is the dispersion coefficient, α_1 parametrizes the self-focusing in birefringent media, and α_2 (called cross-modulation parameter) governs the magnitude of nonlinear coupling between the equations. When $\alpha_2 = 0$, no nonlinear coupling is present despite the fact that "cross-terms" proportional to α_1 appear in the equations [14,27,28]. In this particular case $\psi = \cos(\theta)\chi$, $\phi = \sin(\theta)\chi$, where χ is the solution of a single NLSE with nonlinearity coefficient $\sqrt{2}\alpha_1$.

In the present paper we concern ourselves with the soliton solutions which are localized envelops on a propagating carrier wave. This defines the type of the initial conditions to be used. The general form of the latter is

$$\chi(x, t; X, c, n_\chi) = A_\chi(x + X - ct)\exp\left\{\mathrm{i}\left[n_\chi t - \frac{1}{2}c(x - X - ct) + \delta_\chi\right]\right\}, \tag{2}$$

where χ stands for either ψ or ϕ; c is the phase speed of the envelop, X is the initial position of the center of the soliton; $\vec{n}(n_\psi, n_\phi)$ is the vector of carrier frequencies of the components; $\vec{\delta}(\delta_\psi, \delta_\phi)$ is phase vector of the two components. Note that the phase speed is the same for the two components ψ and ϕ. If they propagate with different phase speeds, after sometime the two components will be in two different positions in space, and will no longer form a single structure. At the same time, the carrier frequencies of the different components can be different, i.e., $n_\psi \neq n_\phi$. In such a case one is faced with the so-called elliptically polarized solitons of SCNLSE, and the two components of the envelop are governed by the following coupled system of differential equations (see [20]):

$$A_\psi'' + (n_\psi + \frac{1}{4}c^2)A_\psi + \left[\alpha_1 A_\psi^2 + (\alpha_1 + 2\alpha_2)A_\phi^2\right] A_\psi = 0, \tag{3a}$$

$$A_\phi'' + (n_\phi + \frac{1}{4}c^2)A_\phi + \left[\alpha_1 A_\phi^2 + (\alpha_1 + 2\alpha_2)A_\psi^2\right] A_\phi = 0. \tag{3b}$$

The above formulation includes the particular case of a circular polarization, when carrier frequencies are equal to each other, i.e., $n_\psi = n_\phi$. Then for the envelopes one has $A_\psi = A(x)\cos(\theta)$, $A_\phi = A(x)\sin(\theta)$, ($\theta \equiv \arctan(\max|\phi|/\max|\psi|)$). When $\theta = 0°$ or $\theta = 90°$, the circular polarization reduces to the so-called linear polarization, in which one of the components is identically equal to zero.

Obviously, the trivial solution of Eq. (3) always persists for asymptotic boundary conditions, and the nontrivial (if exists) is the result of a bifurcation. The problems associated with the bifurcation are elucidated in [20]. We provided another numerical solution to Eq. (3) in [25], which can be readily used as an initial condition for the unsteady computations.

Before turning to the numerical investigation we mention here that the system Eq. (1) possesses three conservation laws when asymptotic boundary conditions are imposed, namely when $\psi, \phi \to 0$ for $x \to \pm\infty$. Following [5,19] we define "mass", M, (pseudo)momentum, P, and energy, E as follows

$$M \stackrel{\text{def}}{=} \frac{1}{2\beta}\int_{-L_1}^{L_2} \left(|\psi|^2 + |\phi|^2\right) \mathrm{d}x, \quad P \stackrel{\text{def}}{=} -\int_{-L_1}^{L_2} \mathcal{I}(\psi\bar{\psi}_x + \phi\bar{\phi}_x)\mathrm{d}x, \quad E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \mathcal{H}\mathrm{d}x, \tag{4}$$

where

$$\mathcal{H} \stackrel{\text{def}}{=} \beta\left(|\psi_x|^2 + |\phi_x|^2\right) - \frac{\alpha_1}{2}(|\psi|^4 + |\phi|^4) - (\alpha_1 + 2\alpha_2)\left(|\phi|^2|\psi|^2\right)$$

is the Hamiltonian density of the system. Here $-L_1$ and L_2 are the left end and the right end of the interval under consideration. The following conservation/balance laws hold, namely

$$\frac{dM}{dt} = 0, \qquad \frac{dP}{dt} = \mathcal{H}|_{x=L_2} - \mathcal{H}|_{x=-L_1}, \qquad \frac{dE}{dt} = 0, \tag{5}$$

which means that for asymptotic boundary conditions the SCNLSE admits at most 3 conservation laws, i.e., the system (1) is non-fully integrable.

3. Numerical method

In order to be able to obtain reliable results for the time evolution of the solution, one needs to devise a difference scheme that represent faithfully the above conservation laws. Such a scheme was proposed in [5], and applied in [19]. This scheme was based on a fast Gaussian elimination solver for multi-diagonal systems [4]. Consequently, the above mentioned scheme was implemented for complex arithmetic in [22], where a complex arithmetic algorithms was developed to generalize the one from [4]. The complex-arithmetic algorithm is four times faster, and we will use it also in the present paper. Thus, for solving Eq. (1) with the initial conditions (2), (3) numerically, we use an implicit conservative scheme in complex arithmetic:

$$\begin{aligned} &\mathrm{i}\frac{\psi_i^{n+1} - \psi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n\right) \\ &+\frac{\psi_i^{n+1} + \psi_i^n}{4}\left[\alpha_1\left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2\right) + (\alpha_1 + 2\alpha_2)\left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2\right)\right], \end{aligned} \tag{6a}$$

$$\begin{aligned} &\mathrm{i}\frac{\phi_i^{n+1} - \phi_i^n}{\tau} = \frac{\beta}{2h^2}\left(\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n\right) \\ &+\frac{\phi_i^{n+1} + \phi_i^n}{4}[\alpha_1\left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2\right) + (\alpha_1 + 2\alpha_2)\left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2\right)], \end{aligned} \tag{6b}$$

on mesh (x_i, t^n) with $x_i = -L_1 + i\,\Delta x$, $\Delta x = (L_2 + L_1)/m$, $i = 1, \ldots, m$ and $t^n = n\,\tau$, $n = 0, 1, 2, \ldots$ It is not only convergent (consistent and stable), but also conserves mass and energy, i.e., there exist discrete analogs M^n and E^n, for (4), which arise from the scheme (for details see [5,19,22]).

$$M^n = \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2 \right) = \text{const},$$

$$E^n = \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}{}^n - \psi_i{}^n|^2 + |\phi_{i+1}{}^n - \phi_i{}^n|^2 \right) + \frac{\alpha_1}{4} \left(|\psi_i^n|^4 + |\phi_i^n|^4 \right) + \frac{\alpha_1 + 2\alpha_2}{2} \left(|\psi_i^n|^2 |\phi_i^n|^2 \right) = \text{const},$$

for $n \geq 0$. These values are kept constant by the scheme during the time stepping. The above scheme is of Crank–Nicolson type for the linear terms and we employ internal iteration to achieve implicit approximation of the nonlinear terms, i.e., we use its linearized implementation [5]. In this way the order of approximation of Eq. (6) is $O(\tau^2 + \Delta x^2)$.

The above nonlinear scheme is implemented via Internal Iterations as follows:

$$\begin{aligned} &\mathrm{i}\frac{\psi_i^{n+1,k+1} - \psi_i^n}{\tau} = \frac{\beta}{2h^2}(\psi_{i-1}^{n+1,k+1} - 2\psi_i^{n+1,k+1} + \psi_{i+1}^{n+1,k+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n) \\ &+ \frac{\psi_i^{n+1,k} + \psi_i^n}{4}[\alpha_1(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2) + (\alpha_1 + 2\alpha_2)(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2)], \end{aligned} \tag{7a}$$

$$\begin{aligned} &\mathrm{i}\frac{\phi_i^{n+1,k+1} - \phi_i^n}{\tau} = \frac{\beta}{2h^2}(\phi_{i-1}^{n+1,k+1} - 2\phi_i^{n+1,k+1} + \phi_{i+1}^{n+1,k+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n) \\ &+ \frac{\phi_i^{n+1,k} + \phi_i^n}{4}[\alpha_1(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2) + (\alpha_1 + 2\alpha_2)(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2)]. \end{aligned} \tag{7b}$$

The iterations are repeated (stepping up the index 'k') until convergence is reached. If the internal iterations are convergent, the scheme in full steps Eq. (6) is absolutely stable due to its conservative nature. For not very large time steps the iteration requires less than half a dozen loops to converge. This is a small price to pay having in mind the inextricably coupled five-diagonal complex structure of the matrix.

The time-stepping scheme needs initial conditions of the type of Eq. (2). In the case of circular polarization, the functions A_χ in Eq. (2) are *sech*-es. In the general case of elliptic initial polarization, the carrier frequencies $n_\psi \neq n_\phi$, and we discretize the system Eq. (3) in the same manner as the evolutionary system, and use Newton's method and Hermitian splines to get the solution. The details can be found in [25]. Our result confirms [20] and can be used with confidence as initial conditions for the problem under consideration.

The above presented scheme and algorithm have been verified for different girds and time increments and the approximation has been confirmed.

4. Results and discussion

The phenomenology of the SCNLSE is very rich, and the interaction of the solitons is strongly influenced by all different parameters: nonlinearity, cross-modulation, carrier frequencies, phase speed, initial polarization of the solitons. In this paper we have set the goal to understand better the polarization dynamics.

4.1. Manakov solitons ($\theta = 45^\circ$). The role of the phase

We begin with the case when there is no cross-modulation, i.e., $\alpha_2 = 0$. This case is alternatively known as Gross-Pitaevskii or Manakov system. The solution was found by Manakov [14] through reducing the system to a single NLSE by setting $\psi = \phi$. In our notations, this corresponds to the case $\theta = 45^\circ$. Actually, the fact that $\alpha_2 = 0$, allows the Manakov solution to exist for any θ, and the shape functions of the envelops are the same *sech*-es. Yet, the system in this case is still nonlinear, and it is very important to investigate numerically the collisions of the *sech*-solitons with the goal to examine if the Manakov solution is robust and if it survives the interaction. To this end we performed several numerical experiments with different initial phase speeds of the solitons, and various phases. Changing the phase speed c, does not seem to lead to qualitative differences of the dynamics, and we present here the results for one representative case $c_l = -c_r = 1$ (the subscripts 'l' and 'r' refer to the left and right soliton, respectively). Similarly, the effect of α_1 is straightforward, and we will discus the case of $\alpha_1 = 0.75$. We select the two components to have the same carrier frequency $n_{l\psi} = n_{r\psi} = n_{l\phi} = n_{r\phi} = -1.5$ and focus on the effect of the initial phases. We treat two different cases: (a) $\delta_l = \delta_r = 0°$ and (b) $\delta_l = 0°$, $\delta_r = 45°$. The results are presented in Fig. 1.

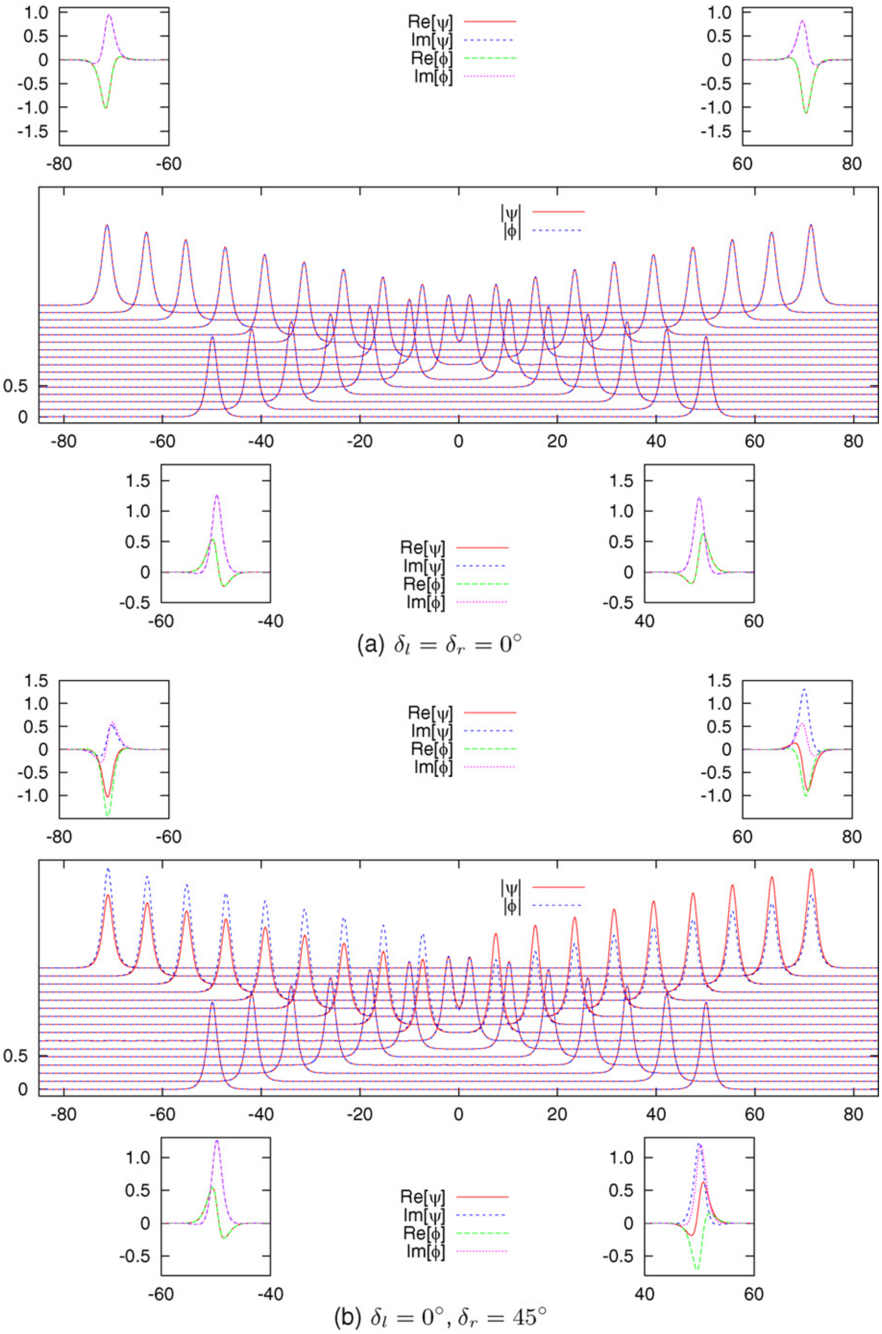
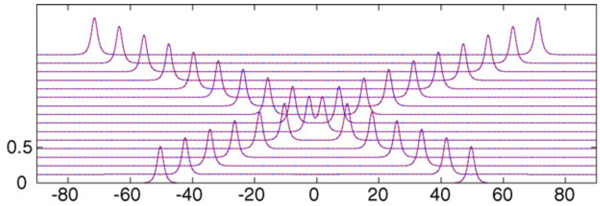


Fig. 1. The role of the initial phases on the interaction of initial circularly polarized solitons with $\theta = 45°$ for $\alpha_2 = 0$, $\alpha_1 = 0.75$, $c_l = -c_r = 1$, $n_{l\psi} = n_{r\psi} = n_{l\phi} = n_{r\phi} = -1.5$.

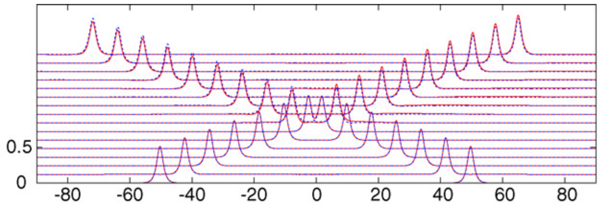
The most important observation is that the Manakov solution is realized only when the initial phases are equal to each other. When one of the phases differs from the other by 45°, the two initially circularly polarized solitons emerge after the interaction as elliptically polarized solitons with $\theta = 36°17'$ and $\theta = 53°37'$, respectively. This result is very significant, and it points out towards a bifurcation. The solution presented in Fig. 1 coexists with the Manakov solution (the latter is valid for any value of the initial phases). Apparently the solution obtained here is robust, while Manakov is not.

Table 1
Polarization evolution during collisions.

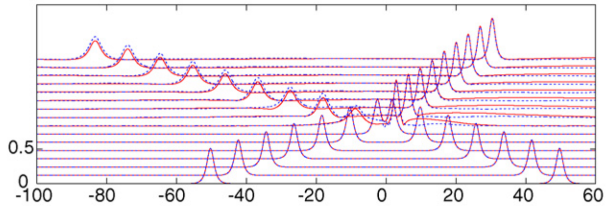
$[\![\delta]\!]$	θ_l^i	θ_r^i	$\theta_l^i+\theta_r^i$	m_ψ^i	m_ϕ^i	p^i	e^i	α_2
(a) Initial values								
0°	45°	45°	90°	2.9837653	2.9790827	−0.11139648E−10	−2.0016750	0
45°	45°	45°	90°	2.9837653	2.9790827	−0.11139716E−10	−2.0016750	0
0°	45°	45°	90°	0.46976218	0.46976218	0.24620106E−10	−0.31538994	4
45°	45°	45°	90°	0.46976218	0.46976218	0.24620078E−10	−0.31538994	4
270°	45°	45°	90°	0.46976218	0.46976218	0.24619929E−10	−0.31538994	4
0°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61555635E−11	−0.25915818	2
45°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61556880E−11	−0.25915818	2
90°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61556301E−11	−0.25915818	2
135°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61557896E−11	−0.25915818	2
140°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61557896E−11	−0.25915818	2
160°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61557427E−11	−0.25915818	2
180°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61556354E−11	−0.25915818	2
270°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61556354E−11	−0.25915818	2
305°	50°08′	50°08′	100°16′	0.58533758	0.92145185	0.61556354E−11	−0.25915818	2
0°	39°51′	50°08′	89°59′	0.75339470	0.75339471	−0.26425613E−07	−0.25915813	2
45°	39°51′	50°08′	89°59′	0.75339470	0.75339471	−0.26425613E−07	−0.25915813	2
135°	39°51	50°08	89°59′	0.75339470	0.75339471	−0.26425613E−07	−0.25915813	2
0°	45°	50°08′	95°08′	0.69922661	0.86728375	0.59265539E−01	−0.40253479	2
45°	45°	50°08′	95°08′	0.69922661	0.86728375	0.59265539E−01	−0.40253479	2
135°	45°	50°08′	95°08′	0.69922661	0.86728375	0.59265539E−01	−0.40253479	2
$[\![\delta]\!]$	θ_l^f	θ_r^f	$\theta_l^f+\theta_r^f$	m_ψ^f	m_ϕ^f	p^f	e^f	α_2
(b) Values after the interaction								
0°	45°	45°	90°	2.9837653	2.9790827	−0.14031559E−02	−2.0016748	0
45°	36°17′	53°37′	89°54′	2.9837653	2.9790827	0.35851357E−02	−2.0016748	0
0°	45°	45°	90°	0.46976218	0.46976218	0.66569146E−03	−0.31538991	4
45°	42°38′	47°49′	90°27′	0.46976218	0.46976218	0.83149016E−03	−0.31538097	4
270°	47°50′	43°09′	90°59′	0.46976218	0.46976218	−0.69621695E−03	−0.31535741	4
0°	48°59′	49°05′	98°04′	0.58533758	0.92145185	0.14500572E−04	−0.25895854	2
45°	49°07′	49°17′	98°14′	0.58533758	0.92145185	0.19469770E−04	−0.25902622	2
90°	55°19′	45°41′	101°0′	0.58533758	0.92145185	0.50089322E−03	−0.25910314	2
135°	58°45′	44°56′	103°41′	0.58533758	0.92145185	−0.21591801E−03	−0.25913752	2
140°	59°57′	44°54′	104°51′	0.58533758	0.92145185	−0.87138012E−04	−0.25914128	2
160°	39°45′	57°37′	97°52′	0.58533758	0.92145185	0.51123279E−03	−0.25915436	2
180°	51°25′	48°25′	99°50′	0.58533758	0.92145185	−0.67917481E−04	−0.25915805	2
270°	44°50′	56°30′	101°20′	0.58533758	0.92145185	−0.42974918E−03	−0.25912060	2
305°	48°54′	50°32′	99°26′	0.58533758	0.92145185	−0.46829621E−03	−0.25907364	2
0°	41°37′	50°42′	92°29′	0.75339470	0.75339471	−0.46310191E−04	−0.25915676	2
45°	37°49′	52°11′	90°	0.75339470	0.75339471	0.68540316E−03	−0.25915751	2
135°	46°32′	43°23′	89°58′	0.75339470	0.75339471	0.70073356E−03	−0.25915432	2
0°	42°11′	52°53′	95°04′	0.69922661	0.86728375	0.58914178E−01	−0.40253371	2
45°	45°04′	49°40′	94°44′	0.69922661	0.86728375	0.60009911E−01	−0.40253265	2
135°	48°47′	46°2′	94°49′	0.69922661	0.86728375	0.59821246E−01	−0.40253466	2



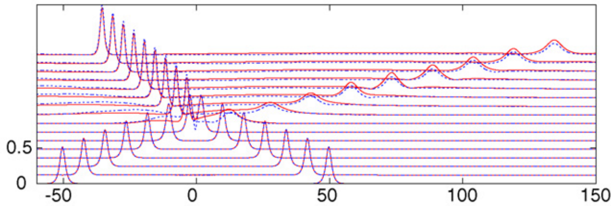
(a) $\delta_l = \delta_r = 0°$



(b) $\delta_l = 0°$, $\delta_r = 45°$



(c) $\delta_l = 0°$, $\delta_r = 90°$



(d) $\delta_l = 0°$, $\delta_r = 135°$

Fig. 2. Initial circular polarization $\theta_l = \theta_r = 45°$ for $\alpha_2 = 4$.

The ‘fragility’ of Manakov solitons was first addressed in [24], where an in-depth study was conducted for various initial configurations for the soliton parameters. The summary of those results is presented in Table 1. We have found as well that the phases of the components play an essential role when the system ‘chooses’ one or another polarization angle of the one-soliton solutions that break away from the site of collision. As a rule the initial polarization is not retained by the solution and outside the cross-section of the interaction, a discontinuity is observed of the polarization which can be called a ‘polarization shock’ and it takes place during the interaction despite the ingoing and outgoing functions are smooth. For nonzero phase one can see that after the interaction, the two QPs become different Manakov solitons than the original two that entered the collision. Apparently the specific choice of outgoing polarization angle is controlled by the magnitude of phase difference $[\![\delta]\!] \equiv \delta_r - \delta_l$. The comparison with the relevant case (with analytic initial condition) considered in [24] enables us to conclude that the envelopes in this case are *sech*-like functions.

In the cited work [24], an important fact of the interaction of polarized vector solitons was unearthed, namely the approximate law of conservation of the total polarization of the system of solitons. Since the conservation was beyond doubt for initially circularly polarized solitons, it seemed important to investigate deeper the issue, and to investigate the polarization dynamics when the initial polarization is not restricted to the circular case.

Now we can move to the case of nontrivial cross-modulation parameter. A very important observation was made in [24], namely that a Manakov type of solution can also be found analytically for nontrivial cross-modulations, but only for one single value of the polarization: $\theta = 45°$. The question arises of what kind of evolution this soliton undergoes during the interaction with another soliton of the same type. For definiteness, we choose $\alpha_2 = 4$. Fig. 2 shows the result for this case.

In order to test and validate the algorithm for computing the initial condition, here we use the numerical solution rather than the analytical *sech*-solution for A_ψ and A_ϕ. The results are indistinguishable from those obtained with the analytical initial condition. It is interesting to mention here that the interaction is relatively robust with respect to changes of the initial phase in the interval $0° \le \delta \le 50°$. Outside of that interval the impact of the initial phase is very strong. Not just the polarization suffers a jump (as already observed in the plain Manakov case), but the phase speeds of the outgoing solitons are radically changed after the interaction. The sensitivity to the values of the initial phases is very strong in the interval $[\![\delta]\!] \in [90°, 180°]$. As can be seen in Fig. 2(c) and (d), the change of the initial polarization difference from 90° to 135° changes the outgoing configuration radically. While for $[\![\delta]\!] = 90°$ the left going soliton acquires a larger phase speed, in the case $[\![\delta]\!] = 135°$, it is the right-going soliton that is sped up significantly. Respectively, their counterparts are slowed down. This effect is in accordance with the laws of conservation of mass and momentum.

4.2. Cases with elliptically polarized solitons as initial conditions

We chose $\alpha_2 = 2$ and settle on the following equal initial polarizations $\theta_l = \theta_r = 50°08'$, which are the result of the different carrier frequencies of the different components of the initial solitons, namely $n_{l\psi} = n_{r\psi} = -1.5$, $n_{l\phi} = n_{r\phi} = -1.1$, $c_l = -c_r = 1$, $c_l = -c_r = 1$, $\alpha_1 = 0.75$, $\alpha_2 = 2$ and focus again on the effect $[\![\delta]\!]$. In Fig. 3 are presented the results of the interaction for different initial phases.

One can see that for $[\![\delta]\!] < 90°$, the interaction is virtually elastic, and even the polarization of each soliton is barely changed. When $[\![\delta]\!] > 90°$, the result changes dramatically: the polarization of the left in-going soliton is larger than the polarization of the right outgoing soliton. Once again, a bifurcation takes place and after the interaction the polarizations are appreciably different from the initial one. Because of the symmetry of the problem, one should expect to encounter both kind of patterns presented in Fig. 3 and their mirror images (horizontally flipped). The choice of the particular configuration is made by the round-off properties of the algorithm, as it is usually the case with bifurcations. In order to make sure that there is no some persistent small inaccuracy of the algorithm that leads to the selection, we took the results for the last two time stages, flipped them horizontally and pugged them in the difference scheme. We found that the residue was of order 10^{-6} which is much better than the truncation error of the scheme. This proves that no approximation error is present in the scheme, even a very insignificant one. In case the residue is of order of the truncation error, this might mean that the truncation error of the implemented algorithm differs from the theoretical truncation error of the scheme. Clearly, this is not the case in the presented algorithm.

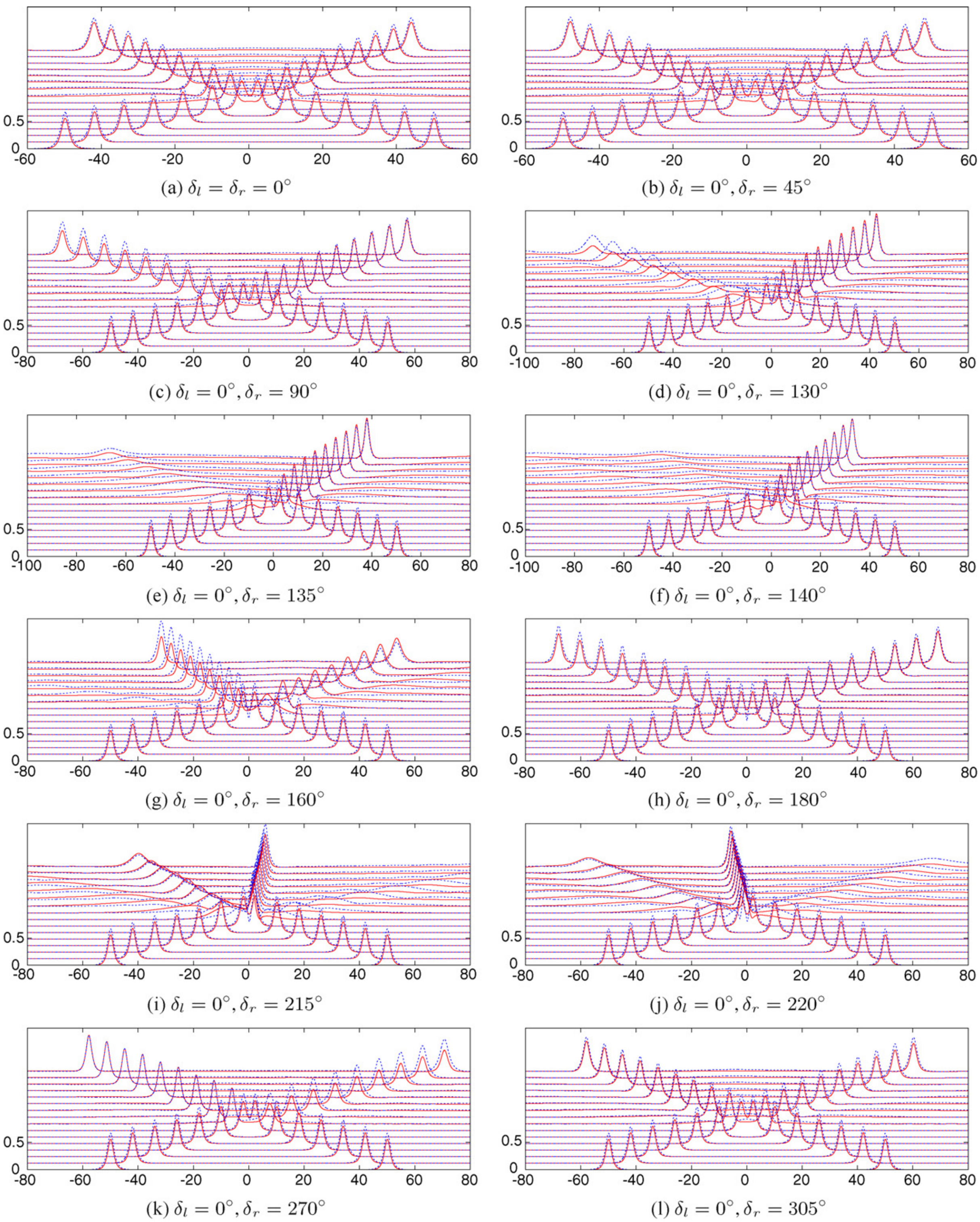
A very interesting, and rather nonintuitive effect is observed for $[\![\delta]\!] \in [130°, 140°]$ in Fig. 3(e) and (f): one of the QPs loses significant portion of its ‘mass’ (as testified by the decreased amplitude) contributing it to the other QP. It still carries appreciable momentum because of its increased phase speed. The other outgoing soliton has larger mass but smaller phase speed. This effect is not exactly what is called ‘trapping’, because the momentum and the energy of one of the solitons is not entirely transferred to the other, but it seems akin to trapping as far as the diminished amplitude of one of the solitons is concerned. It goes beyond the scope of the present work to dwell deeper on the ‘trapping’ for the system under consideration, and it will be done elsewhere.

For $[\![\delta]\!] \geq 160°$ the outcome of the interaction is radically different. Now, the left going soliton has a large amplitude and is relatively slower (note that the right-going soliton has a slower phase speed than the initial phase speed). Now the polarization of the left-going soliton is also significantly larger than for the other cases. Increasing further $[\![\delta]\!]$ to 180°, does not change qualitative the interaction, but merely mitigates the effect of the changes. The intuitive expectation is that $[\![\delta]\!] = 180°$ may have the same quantitative effect as $[\![\delta]\!] = 0°$ is not confirmed. Clearly, if the two components of the initial vector solitons are in anti-phase is not equivalent to the case when they are in phase ($[\![\delta]\!] = 0°$).

For this reason we investigated also the interaction in the interval $[\![\delta]\!] \in (180°, 360°)$, and the results are presented in the respective panels of Fig. 3. We have discovered another very sensitive interval $[\![\delta]\!] \in (210°, 230°)$, where one of the solitons becomes much steeper, and a third very fast soliton with relatively small amplitude is born whose momentum balances the momenta of the two large solitons escaping to the left (see Fig. 3(j) and (i), to a smaller extent). The rest of the values $270° < [\![\delta]\!] < 360°$ do not lead to such violent changes, and the effect is confined to some moderate changes of the soliton polarizations.

Similarly to the circularly polarized solitons, the sum of polarizations of the outgoing solitons is approximately equal to the sum of the original ones. Thus one can once again argue the case for ‘conservation’ of the polarization. The numerical values are presented in Table 1. The next experiment is to investigate the case when the two initial solitons have elliptic polarization and these polarizations are complementary to each other. We chose one of the solitons to have the same parameters as the previous case, and take the other one to have the “complementary” polarization. To construct such a configuration we take $n_{l\psi} = n_{r\phi} = -1.1$, $n_{l\phi} = n_{r\psi} = -1.5$. We keep all the other parameters the same: $c_l = -c_r = 1$, $\alpha_1 = 0.75$, $\alpha_2 = 2$. For the initial polarization we obtain: $\theta_l = 39°51'$, $\theta_r = 50°08'$, $\theta_l^i + \theta_r^i \approx 90°$.

The results are presented in Fig. 4 for four different values of $[\![\delta]\!]$. The results can be summarized as follows: For $[\![\delta]\!] = 0°$ (Fig. 4(a)), the polarizations are virtually the same as for the initial configuration, with a slight tendency to diminishing the difference between them (see Table 1). For $[\![\delta]\!] = 45°$ (Fig. 4(b)), the polarizations are close to the initial configuration, with a slight tendency to increasing the difference between them (see Table 1), which is qualitative difference from the case $[\![\delta]\!] = 0°$. There is no much difference between the case $[\![\delta]\!] = 90°$ (Fig. 4(c)) and $[\![\delta]\!] = 45°$ (Fig. 4(b)). Finally, for $[\![\delta]\!] = 135°$ (Fig. 4(d)) the tendency is reversed back to diminishing the difference between the outgoing polarizations.



(a) $\delta_l = \delta_r = 0^\circ$

(b) $\delta_l = 0^\circ, \delta_r = 45^\circ$

(c) $\delta_l = 0^\circ, \delta_r = 90^\circ$

(d) $\delta_l = 0^\circ, \delta_r = 130^\circ$

(e) $\delta_l = 0^\circ, \delta_r = 135^\circ$

(f) $\delta_l = 0^\circ, \delta_r = 140^\circ$

(g) $\delta_l = 0^\circ, \delta_r = 160^\circ$

(h) $\delta_l = 0^\circ, \delta_r = 180^\circ$

(i) $\delta_l = 0^\circ, \delta_r = 215^\circ$

(j) $\delta_l = 0^\circ, \delta_r = 220^\circ$

(k) $\delta_l = 0^\circ, \delta_r = 270^\circ$

(l) $\delta_l = 0^\circ, \delta_r = 305^\circ$

Fig. 3. $\theta_l = \theta_r = 50^\circ 08'$, $\alpha_2 = 2$.

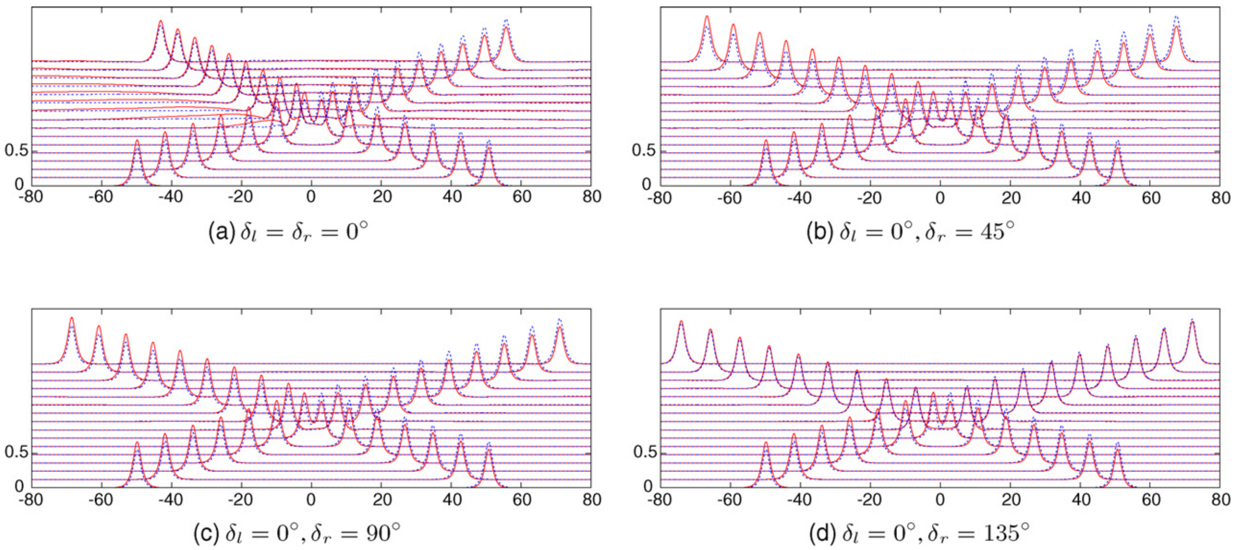


Fig. 4. $n_{l\psi} = n_{r\phi} = -1.1$, $n_{l\phi} = n_{r\psi} = -1.5(\theta_l = 39°51', \theta_r = 50°08')$ for $\alpha_2 = 2$.

In all cases presented in Fig. 4, a "slight" polarization shock is experienced after the interaction of the solitons, but once again an approximate conservation of the total polarization is observed (Table 1).

The table also gives info on the 'masses' of the solitons, pseudomomentum, and total energy of the system of solitons. On can see the perfect conservation of the mass and the acceptable conservation of energy. The pseudomomentum is conserved within the truncation order of the scheme. For all the cases considered in the paper the total pseudomomentum is expected to be equal to zero due to symmetry. It is to be noted here that the exact conservation of the pseudomomentum is to be expected only for asymptotic boundary conditions. Even when the computational box is large enough, there are some nontrivial values of the derivatives of the sought function near the boundaries where the functions themselves are kept equal to zero. These small values are of order of the truncation error, and they contribute to the small deviation of the computed pseudomomentum from zero.

5. Conclusion

In the present paper the polarization dynamics of interacting vector solitons of the Coupled Nonlinear Schrödinger Equations (SCNLSE) is investigated numerically by means of an energy conserving difference scheme. The initial condition is obtained numerically by solving the system of ODEs for the shape of the stationary propagating vector soliton. For the latter the carrier frequencies of the different components of the vector soliton are different and the amplitude of the envelop is no longer a *sech*. Different combinations of elliptically polarized (e.g., circularly polarized for the particular case of angle of polarization 45°) solitons are used as the initial condition and the evolution of the system is followed numerically. Using a general initial polarization enriches significantly the observed effects of the nonlinear coupling in SCNLSE. The evolution of the polarization during the collision of elliptically polarized solitons is the object of the present paper. Our results show that the polarization dynamics turns out to be very susceptible to the initial phases of the solitons, which appears to be a novel result, not known from the literature.

First, we have investigated the case of trivial cross-modulation, $\alpha_2 = 0$. For this case, the Manakov analytical solution is available. Our computations show that there exist initial phase for which the Manakov solutions persists after the interaction, but there are also values of the phases for which the polarization changes to elliptic after the interaction. This means that a bifurcation takes place, and along with the solution with constant polarization (the analytical Manakov solution), another solution appears for which a kind of jump (or 'shock') of the polarization takes place during the cross-section of the interaction. This result is very important to put the Manaklov solution in the proper perspective.

Second, we treated cases with nontrivial cross-modulation, $\alpha_2 \neq 0$. We found that the actual magnitude of α_2 has only quantitative effect on the results, and the qualitative effect is present even for $\alpha_2 < 1$. Our results show that the initially circular polarization inevitably transforms to elliptic polarization of the outgoing solitons. What is more important is that the nontrivial cross-modulation leads to increase or decrease of the phase speed of the outgoing solitons, which also brings about drastic changes of their amplitudes. In some cases the amplitude is so small that the result can be classified as ‘trapping’ in the sense that one of the outgoing solitons captures the predominant energy of the process, while the other appears more as a small disturbance. The effects scale with α_2, in the sense that for larger α_2 the mentioned deviations are bigger. Thus, one of the main results of the present paper is in identifying the role of the initial phase, showing that it can have a dramatic effect on the interaction for nontrivial cross-modulations.

In all considered cases we found that although the initial polarizations can suffer a shock (or a jump) during the interaction, the total polarization is fairly well conserved. The actual polarization angles are strongly influenced by the initial phases, but their sum is virtually independent of the phase.

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