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2011 J. Phys. A: Math. Theor. 44 112001

(http://iopscience.iop.org/1751-8121/44/11/112001)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **44** (2011) 112001 (10pp) doi:10.1088/1751-8113/44/11/112001

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An improved formula for the frequency shift due to a variable phase speed

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Received 10 May 2010, in final form 14 September 2010
Published 23 February 2011
Online at stacks.iop.org/JPhysA/44/112001

Abstract
The mathematical problem of linear wave propagation in a medium with slowly spatially-varying phase speed is examined by means of the multiple-scale perturbation technique. Specifically, the propagation of electromagnetic waves in a gravitational field, when the speed of light is a function of the strength of the potential, is considered. A new solution is found, which improves the well-known formula, based on a 'quasi-constant' assumption. The latter is compared to the new expression for the gravitational redshift of light emitted from the surface of a massive body, and it is shown to deviate up to 20% in the region around a few radii from the emitting surface.

PACS numbers: 02.30.Mv, 41.20.Jb, 95.30.Sf

1. Introduction

The propagation of linear waves through a medium with constant phase speed is governed by a second-order hyperbolic partial differential equation (PDE) with constant coefficients. In this case, the relationship between the frequency and the wavenumber is given by an algebraic dispersion relation [1, section 4.7]. A wave emitted at a given point in space propagates unchanged (in the absence of dissipation and dispersion), and measuring its temporal frequency at any other point in space yields the same value. When the phase speed is not constant, but a function of the spatial variable, the dispersion relation does not have to be an algebraic (polynomial) equation. It may happen that solving this dispersion relation is as difficult as solving the original hyperbolic PDE. However, in the case of slow (spatial) variation of the phase speed (i.e. the PDE's coefficients), an approximate solution can be obtained by perturbation methods. The small parameter in this case is the ratio of the effective length scale on which the phase speed varies to the (characteristic) wavelength of the propagating wave.

When the properties of the medium—specifically, the phase speed of waves—change on a much longer scale than the length scale of the propagating wave, it is tempting to consider, locally, a PDE with constant coefficients, acknowledging the variability only when detecting the wave at a different spatial location. In other words, at the receiver a frequency $\omega_{\mathrm{rec}} \approx kc(\vec{x}_{\mathrm{rec}})$ different from the one at the emitter $\omega_{\mathrm{em}} \approx kc(\vec{x}_{\mathrm{em}})$ is predicted, under the assumption that the wavenumber k is the same constant everywhere. Such an approach can be called 'quasi-constant' (in analogy to processes where the time derivative in the equations is neglected, leaving time as a parameter, which are called 'quasi-stationary' or 'quasi-static'). This approximation disregards the gradual changes of the frequency and wavenumber with respect to the spatial variable. In this work, we retain these, thus accounting for the *cumulative* effect of the variable phase speed on waves propagating through this medium.

Although the mathematical analysis presented below is applicable to any linear wave phenomenon (e.g. acoustic waves in fluids, small flexural deformations of solids, classical water waves, etc [1]), we are interested in the propagation of electromagnetic waves (e.g. light). Einstein [2, 3] proposed that the speed of light is influenced by the gravitational field in the local patch of spacetime where it propagates. Specifically, the speed of light becomes a function of the distance from the source of gravitation (see, e.g. [4, chapter 16], or [5, section 3.3.2], for a modern derivation based on general relativity). An immediate consequence is that, in certain spacetime geometries, the path that the light ray follows, which is defined as a null geodesic on the Riemannian manifold modeling the spacetime [4, chapter 10], may be curved [5, section 6.2]. This is the essence of the well-known phenomenon of gravitational lensing [6, 7].

Regardless, the textbook approach (see, e.g. [8, section 5], or [4, chapter 16]) is to then compute the frequency shift by taking the ratio of the phase speeds at the emitter's and receiver's locations only with the *coordinate* time units transformed appropriately according to the curvature of the manifold into *proper* time units ('time dilation'). However, this is once again an application of the quasi-constant light-speed assumption because the speed of light varies continuously along such a curved path, and a wave equation with non-constant coefficients is not generally covariant under the transformation from coordinate to proper time. Therefore, the computed frequency shift is an approximate one *independently* of whether the argument is made using the equivalence principle or the Schwarzschild solution in general relativity. These ubiquitous approaches are only 'valid to a first approximation' [3, page 105]. Hence, a rigorous investigation of the frequency shift of monochromatic electromagnetic waves propagating through a spacetime with spatially-variable speed of light is lacking in the literature. Only for certain very specific functional forms of the variability, some exact results for traveling waves have been obtained [9].

Here, we do not dwell on the various nuances of how to apply and motivate the quasi-constant approximation. Instead, we focus on the variable-coefficient evolution equation for an electromagnetic wave. To this end, we present an asymptotic solution based on the method of multiple scales (MS), for the case when the dependence of the phase speed on the radial spatial coordinate is 'slow,' to account for the change of the frequency along the curved path of the light ray. In other words, we extend the one-length-scale textbook approach based on geometric optics to a more general (and free of secular terms) two-length-scale analysis as suggested in [10, page 572]. We show that, depending on the spatial scales involved, the formula based on the quasi-constant assumption can significantly overestimate the red- and/or blueshift of light.

2. Position of the problem

It can be shown that the propagation of transverse electromagnetic waves emitted from the (effective) surface of some spherical body (e.g. a star) is governed by a radially-symmetric wave equation in spherical coordinates (see (A.12)):

$$\frac{\partial^2 Q}{\partial t^2} = c^2(r)\frac{\partial^2 Q}{\partial r^2} + \alpha c(r)c'(r)\frac{\partial Q}{\partial r} - \frac{\alpha}{r}c(r)c'(r)Q, \qquad \alpha \in \{1, 2\}, \tag{1}$$

where Q is related to a transverse component of the electromagnetic field, r is the radial coordinate measured from the center of the spherical body and c is the speed of light. The variability of the speed of light can be due to the material properties of the medium in which the electromagnetic wave propagates as is the case for, e.g. photonic crystals [11] ($\alpha = 2$), or the variability can result from a spatially-varying fundamental tensor due to the influence of gravitation [2, 3, 12–14] ($\alpha = 1$).

According to the model of Einstein [2, 3], the speed of light c depends on the radial coordinate r through the gravitational potential Φ as follows:

$$c = c(r) = c_0\left(1 + \frac{\Phi(r)}{c_0^2}\right), \qquad \Phi(r) = -\frac{GM}{r}, \tag{2}$$

where $G = 6.67300 \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$ is the gravitational constant, M is the mass of the emitting body centered at the origin of the coordinate system (e.g. for star such as the Sun $M = M_\odot = 1.98892 \times 10^{30}$ kg), and c_0 is the absolute speed of light far from the source of gravity. For estimates of the size of the parameters, we can take this to be the value at the surface of the Earth, i.e. 299, 792, 458 $\mathrm{m\,s^{-1}}$, to a good approximation.

The original formula of Einstein was refined by acknowledging the precise effect of the spatially-varying fundamental tensor (see, e.g. [4, 5, 8]), which leads to the general-relativistic formula

$$c(r) = c_0\left(1 + 2\frac{\Phi(r)}{c_0^2}\right) = c_0\left(1 - \frac{\delta}{r/r_0}\right), \qquad \delta := \frac{2GM}{c_0^2 r_0}. \tag{3}$$

It differs from (2) by the factor of 2 in front of Φ. Here, r_0 is the radius of the (spherical) object emitting the electromagnetic waves, i.e. the location of the surface from which the waves are emitted. The parameter δ is the ratio of the Schwarzschild radius to the actual radius of the emitting body (the 'deviation from a flat spacetime' [5, section 6.1.2]), so it must be such that $\delta \leqslant 1$.

For example, the radius of the Sun is $r_0 = R_\odot = 6.95500 \times 10^8$ m, which gives $\delta = 4.246\,49 \times 10^{-6} \ll 1$. For white dwarfs, such as Sirius B, this parameter can be up to two orders of magnitude larger (see, e.g. [15]), but it is much smaller for large stars such as Betelgeuse, Rigel, Mira, etc. In obtaining the perturbative solution, the smallness of δ is *not* essential, meaning the result is valid even for the potentials created by very massive objects (e.g. neutron stars and objects close to the black-hole stage). However, to obtain simple formulas for the redshift and blueshift, it is convenient to assume $\delta \ll 1$, and, in practice, it is safe to take $\delta \simeq 10^{-4}$.

Suppose ω_0 is a representative frequency for the electromagnetic wave under consideration, e.g. the central frequency of the wave packet representing a particular spectral line. The typical scale for ω_0 is on the order of a few to a few hundreds of THz for waves in the infrared to visible spectrum. Then, we introduce the dimensionless temporal and spatial variables

$$\tau = \omega_0 t, \qquad x = (r - r_0)\omega_0/c_0. \tag{4}$$

We can write $x = \epsilon^{-1}(r/r_0 - 1) \Leftrightarrow r = r_0(1 + \epsilon x)$, where the (small) parameter $\epsilon := c_0/(r_0\omega_0)$ is the ratio between the length scale c_0/ω_0 set by the frequency and phase speed of the wave and the characteristic length scale r_0 of the emitting body. The limit $\epsilon \ll 1$ corresponds to the case when the change in $c(r)$ over a spatial period of the wave is small. This leads to the natural introduction of a 'long' spatial variable $x_1 = \epsilon x$.

Rewriting (3) in terms of this notation, we have

$$c(r) = c_0\hat{c}(x_1), \qquad \hat{c}(x_1) := 1 - \frac{\delta}{1+x_1}. \tag{5}$$

Accordingly, the partial derivatives transform as

$$\frac{\partial}{\partial t} = \omega_0\frac{\partial}{\partial \tau}, \qquad \frac{\partial}{\partial r} = \frac{1}{r_0\epsilon}\frac{\partial}{\partial x}. \tag{6}$$

Introducing (4), (5) and (6) into (1), the governing PDE takes the following dimensionless form:

$$\frac{\partial^2 q}{\partial \tau^2} = \hat{c}^2(x_1)\frac{\partial^2 q}{\partial x^2} + \alpha\epsilon\hat{c}(x_1)\hat{c}'(x_1)\frac{\partial q}{\partial x} - \frac{\alpha\epsilon^2}{1+\epsilon x}\hat{c}(x_1)\hat{c}'(x_1)q, \tag{7}$$

where $q(x,\tau) = Q(r,t)/Q_0$, Q_0 is arbitrary (since the PDE is linear and homogeneous), and a prime denotes differentiation with respect to the argument of the function.

3. Solution with slowly-varying wavenumber and envelope

As we discussed above, the problem has a 'short' spatial scale x and a 'long' spatial scale $x_1 = \epsilon x$. Hence, it is an ideal candidate for the perturbative method of MS [16, 17]. The standard MS approach to a PDE with variable coefficients, such as (7), is presented in Holmes' book [16, section 3.6]. Here, we take the generalized MS approach [17, section 6.4], which is similar to the WKB and Krylov–Bogolyubov averaging methods for ODEs [18]. We seek a slowly-varying outgoing plane wave solution with harmonic time dependence, so we can skip the introduction of a 'short' time $\tau_1 = \epsilon\tau$ and proceed directly to the ansatz

$$q(x,\tau) = A(x_1)\exp[\mathrm{i}\tau - \mathrm{i}k(x_1)x], \tag{8}$$

where in addition to the previously-defined variables we now also have the dimensionless wavenumber k. We must now determine A and k.

Under the ansatz in (8) and within $\mathrm{O}(\epsilon^2)$, the derivatives in (7) are

$$\frac{\partial^2 q}{\partial \tau^2} = -A\exp(\mathrm{i}\tau - \mathrm{i}kx), \tag{9a}$$

$$\frac{\partial q}{\partial x} = (\epsilon A' - \mathrm{i}kA - \mathrm{i}x\epsilon k'A)\exp(\mathrm{i}\tau - \mathrm{i}kx), \tag{9b}$$

$$\frac{\partial^2 q}{\partial x^2} = [-k^2A - 2\mathrm{i}\epsilon(kA' + k'A) - 2\epsilon xkk'A]\exp(\mathrm{i}\tau - \mathrm{i}kx), \tag{9c}$$

where, as before, a prime denotes differentiation with respect to the argument (i.e. x_1).

Upon introducing (9a), (9b) and (9c) into (7), we obtain

$$(1 - \hat{c}^2k^2 - 2\epsilon x\hat{c}^2kk')A - 2\mathrm{i}\epsilon\hat{c}^2(kA' + k'A) - \alpha\mathrm{i}\epsilon\hat{c}\hat{c}'kA = 0, \tag{10}$$

which is accurate within $\mathrm{O}(\epsilon^2)$. Recalling that $\epsilon x = x_1$ and separating the real and imaginary parts of this last equation, we arrive at the system

$$1 - \hat{c}^2k^2 - 2x_1\hat{c}^2kk' = 1 - (x_1k^2)'\hat{c}^2 = 0, \tag{11a}$$

$$kA' + k'A + \frac{\alpha\hat{c}'}{2\hat{c}}kA = (kA)' + \frac{\alpha\hat{c}'}{2\hat{c}}kA = 0. \tag{11b}$$

Equation (11a) can be integrated to give

$$x_1k^2(x_1) = \int_0^{x_1}\frac{\mathrm{d}\xi}{\hat{c}^2(\xi)} + \text{const}. \tag{12}$$

When $x_1 = 0$, the left-hand side of the above equation vanishes, so the arbitrary constant on the right-hand side must be equal to zero. Now, we can introduce the expression from (5) in the above integral to get (recalling that $\delta \leqslant 1$)

$$k^2(x_1) = \frac{1}{x_1}\int_0^{x_1}\left(1 - \frac{\delta}{1+\xi}\right)^{-2} \mathrm{d}\xi = \frac{1}{x_1}\int_0^{x_1}\frac{(1+\xi)^2}{(1-\delta+\xi)^2}\,\mathrm{d}\xi$$
$$= 2 + \frac{1}{(1-\delta)x_1} + \frac{(1+x_1)^2}{(\delta-1-x_1)x_1} + \frac{4\delta}{x_1}\tanh^{-1}\left(\frac{x_1}{2-2\delta+x_1}\right). \quad (13)$$

Similarly, (11*b*) can be integrated to yield

$$k(x_1)A(x_1) = \frac{D}{\hat{c}^{\alpha/2}(x_1)}, \quad (14)$$

where D is a constant of integration. Without loss of generality, we can assume $A(0) = 1$; then, taking the limit $x_1 \to 0$ in (13) and acknowledging (5), we arrive at $1/(1-\delta) = D/\hat{c}^{\alpha/2}(0) \Rightarrow D = (1-\delta)^{\alpha/2-1}$. Thus, we have

$$A(x_1) = \frac{(1-\delta)^{\alpha/2-1}}{k(x_1)\hat{c}^{\alpha/2}(x_1)} \quad (15)$$

with $k(x_1)$ and $\hat{c}(x_1)$ as defined through (13) and (5), respectively, and $\alpha \in \{1, 2\}$.

4. The gravitational redshift

Let us further suppose (as argued earlier) that $\delta \ll 1$. Then, we can reduce the integral in (12) to a simpler expression, namely

$$k^2(x_1) \simeq \frac{1}{x_1}\int_0^{x_1} 1 + \frac{2\delta}{1+\xi}\,\mathrm{d}\xi = 1 + \frac{2\delta}{x_1}\ln(1+x_1). \quad (16)$$

Consequently, within the $\mathrm{O}(\delta^2)$ order of approximation, the wavenumber is given by

$$k(x_1) \simeq \sqrt{1 + \frac{2\delta}{x_1}\ln(1+x_1)} \simeq 1 + \frac{\delta}{x_1}\ln(1+x_1). \quad (17)$$

Thus, $k(0) \simeq 1+\delta$ and $k(\infty) = 1$. This means that the wavenumber (respectively wavelength) of the electromagnetic wave is decreased (respectively increased) by a factor of $1+\delta$ as $x_1 \to \infty$. Hence, a redshift will be detected far away from the source, if one knows *a priori* the wavelength of the emitted wave.

Consider the relative change in the wavenumber of the wave as measured at some distance $x_1 = X_1 \gg 1$ away from the emitting surface:

$$\frac{k(X_1) - k(0)}{k(0)} \simeq -\delta\left[1 - \frac{\ln(1+X_1)}{X_1}\right]. \quad (18)$$

The frequency of the wave is $\omega(x_1) = \hat{c}(x_1)k(x_1) \simeq k(x_1)$ to the lowest order in δ so, in the original variables, (18) is

$$\frac{\omega_{\mathrm{rec}} - \omega_{\mathrm{em}}}{\omega_{\mathrm{em}}} \simeq -\delta\left[1 - \frac{\ln(r/r_0)}{r/r_0 - 1}\right], \quad (19)$$

which, we note again, is accurate within $\mathrm{O}(\delta^2)$.

Under the quasi-constant light-speed assumption of Einstein [2, 3], which also takes $\delta \ll 1$, the ratio of the 'final' and 'initial' frequencies is proportional to the negative ratio of

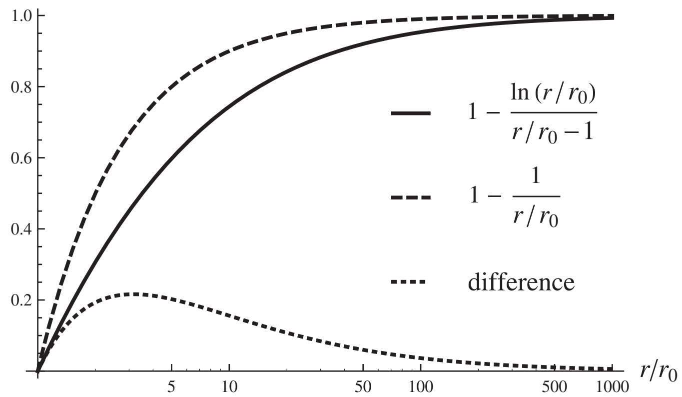

Figure 1. Comparison of the original formula in (20) with the refined formula in (19) for the gravitational redshift. Note that the abscissa is logarithmic.

the values of the speed of light at these two locations, to a first approximation, giving a relative change of

$$\frac{\omega_{\text{rec}} - \omega_{\text{em}}}{\omega_{\text{em}}} \simeq -\frac{c(r) - c(r_0)}{c(r_0)} = -\delta\left(1 - \frac{1}{r/r_0}\right) \tag{20}$$

as in [5, (3.25)]. A graphical comparison between (19) and (20) (both scaled by $-\delta$) is shown in figure 1.

From figure 1 it is clear that the ubiquitous formula predicts a frequency shift persistently higher than the more accurate expression derived here. This means that the former always *overestimates* the gravitational redshift. Although the difference is not very drastic, the proposed formula in (19) is clearly an improvement over the standard result in (20). It is important to note that the biggest deviation from the quasi-constant assumption is at distances on the order of a few radii of the emitting body. For a distance of over one astronomical unit (i.e. $r \gtrsim 215 r_0$ when $r_0 = R_\odot$), the difference between the two formulas is negligible. Similarly, for $r \approx r_0$ the formulas once again agree closely, and the verification of the gravitational redshift from experiments conducted on Earth [19, 20] is not affected by the present discussion. The cumulative effect of the spatially-varying speed of light accounted for in (18) and (19) is also significant for objects with large mass and small radius (e.g. white dwarfs and neutron stars), especially if the redshift is measured close to the emitting surface.

Finally, note that the wavenumber, and consequently the frequency shift, do not depend on the value of $\alpha \in \{1, 2\}$ chosen in the governing wave equation (1), only the amplitude of the wave does.

5. The gravitation blueshift

Now, if we were to consider the *incoming* wave, i.e. change $-\mathrm{i}k(x_1)x$ to $+\mathrm{i}k(x_1)x$ in (8), the above analysis can be redone, without any significant changes, arriving at the analog of (19) with the pre-factor $-\delta$ replaced by $+\delta$. Thus, as with Einstein's original result, if the emitter is far away from the massive body, then an observer at body's surface measures a blueshift.

However, this is only a *qualitative* result because to study the blueshift rigorously we would have to consider the spherical wave emitted by a point source at some $x_1 \gg 1$. Then, the problem can no longer be treated as exactly radially symmetric. Consequently, the shape of path of the detected ray of light would make a difference. For curved spacetime, this path is not the line connecting the emitter with the center of the massive body creating the gravitational field. This leads to questions regarding the bending of light. Strictly speaking, (19) and its blueshift analog apply to spherical waves emitted uniformly from a *surface* of large radius. For electromagnetic waves incoming from a far away *point* source, this is not the case.

Nevertheless, for a qualitative assessment of the change of the frequency experienced by waves incoming toward a source of gravitation, the bulk effect of the blueshift is adequately represented by (19) with the opposite sign.

6. Conclusion

In this work, a rigorous perturbative solution, based on the method of multiple scales, is obtained for the radially-symmetric wave equation in spherical coordinates with spatially-varying phase speed according to Einstein's celebrated hypothesis about the dependence of the speed of light on the gravitational potential. The standard formula assumes that the change of the spatial wavenumber (and, as a result, the temporal frequency at a point) can be obtained by simply inserting the local speed of light in the dispersion relation of the constant-coefficient wave equation. This amounts to a 'quasi-constant' assumption for the phase speed. Alternatively, one can perform a *regular* perturbation expansion [13] to obtain the ubiquitous frequency shift but then secular terms arise. The present result properly accounts for the cumulative change in the phase speed due to gravity as the wave propagates, up to order $O(\epsilon^2, \delta^2)$ in the introduced small parameters.

Our analysis leads to a more precise formula for the redshift and blueshift. In fact, as shown in figure 1, the maximum difference between the two formulas is over 20% at $r \simeq 3r_0$. Consequently, for waves observed within a few radii of the surface of the emitting body, the quasi-static formula significantly overestimates the redshift. Thus, (19) is a notable improvement over the well-known expression for observations near the emitting source and also for bodies with large mass and small radius (i.e. when ϵ is small but not negligible).

Finally, we note that the refined equation for the speed of light in a gravitational field (i.e. (3)) applied to the fine structure constant [21] has been used to suggest an explanation of the Pioneer anomaly [22]. It appears possible that the present results may also be used to shed some light on this issue. Additionally, through an analogy to viscoelastic continua, it has been shown that some of the effects of the redshift can be explained by dispersive dissipation [23]. It would be of interest to extend those results to dispersive wave equations with *variable coefficients* via the present approach.

Appendix. Derivation of the governing wave equation

Maxwell's equations *in vacuo* can be written as

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{1}{\varepsilon}\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{E}\right), \qquad \frac{\partial^2 \vec{B}}{\partial t^2} = -\frac{1}{\mu}\nabla \times \left(\frac{1}{\varepsilon}\nabla \times \vec{B}\right). \tag{A.1}$$

The operator on the right-hand side of the first equation in (A.1) can be manipulated as follows:

$$\begin{aligned}\frac{1}{\varepsilon}\nabla \times \left(\frac{1}{\mu}\nabla \times \vec{E}\right) &= \frac{1}{\varepsilon}\left(\nabla\frac{1}{\mu}\right) \times (\nabla \times \vec{E}) + \frac{1}{\varepsilon\mu}\nabla \times \nabla \times \vec{E} \\ &= \frac{1}{\varepsilon}\left(\nabla\frac{1}{\mu}\right) \cdot (\nabla\vec{E})^\top - \frac{1}{\varepsilon}\left(\nabla\frac{1}{\mu}\right) \cdot (\nabla\vec{E}) - \frac{1}{\varepsilon\mu}\nabla^2\vec{E} \\ &= \frac{1}{\varepsilon}\left(\nabla\frac{1}{\mu}\right) \cdot (\nabla\vec{E})^\top - \frac{1}{\varepsilon}\nabla \cdot \left(\frac{1}{\mu}\nabla\vec{E}\right),\end{aligned} \tag{A.2}$$

where we have used the identities $\vec{\Upsilon} \times (\nabla \times \vec{\Xi}) = \vec{\Upsilon} \cdot (\nabla\vec{\Xi})^\top - \vec{\Upsilon} \cdot (\nabla\vec{\Xi})$ and $\nabla \times \nabla \times \vec{\Xi} = \nabla(\nabla \cdot \vec{\Xi}) - \nabla^2\vec{\Xi}$, and we have acknowledged that $\nabla \cdot \vec{E} = 0$ *in vacuo*. In a similar fashion, we can decompose the operator acting on $\vec{B}$ in the second equation in (A.1) arriving at

$$\begin{aligned} \frac{\partial^2 \vec{E}}{\partial t^2} &= \frac{1}{\varepsilon}\nabla \cdot \left(\frac{1}{\mu}\nabla\vec{E}\right) - \frac{1}{\varepsilon}\left(\nabla\frac{1}{\mu}\right) \cdot (\nabla\vec{E})^\top, \\ \frac{\partial^2 \vec{B}}{\partial t^2} &= \frac{1}{\mu}\nabla \cdot \left(\frac{1}{\varepsilon}\nabla\vec{B}\right) - \frac{1}{\mu}\left(\nabla\frac{1}{\varepsilon}\right) \cdot (\nabla\vec{B})^\top. \end{aligned} \tag{A.3}$$

These equations can be rewritten in terms of the speed of light, $c \equiv 1/\sqrt{\varepsilon\mu}$, but there are three cases to consider. The first two cases are when either the permitivity $\varepsilon = \varepsilon_0 = \text{const}$ or the permeability $\mu = \mu_0 = \text{const}$, while the other may vary with the coordinates (this type of assumption is common in, e.g., photonic crystals [11]). Applying the first and second assumptions to the first and second equations in (A.3), respectively, we arrive at

$$\frac{\partial^2 \vec{P}}{\partial t^2} = \nabla \cdot (c^2\nabla\vec{P}) - (\nabla c^2) \cdot (\nabla\vec{P})^\top, \tag{A.4}$$

where $\vec{P}$ stands for either $\vec{E}$ or $\vec{B}$. The third case is when both the permitivity and the permeability are the same function of the coordinates: $\varepsilon = \varepsilon_0 f(\vec{x})$ and $\mu = \mu_0 f(\vec{x})$. Then, $c = c_0/f(\vec{x})$ and both equations in (A.3) reduce to

$$\frac{\partial^2 \vec{P}}{\partial t^2} = \nabla \cdot (c\nabla\vec{P})c - (\nabla c) \cdot (\nabla\vec{P})^\top c. \tag{A.5}$$

This third case relates to the propagation of light in a gravitational field because the fundamental tensor itself is a function of the spatial coordinates via the gravitational potential. Hence, the same variable coefficient will enter the formulas for the covariant derivatives, leading to the case of (A.5). (For derivations of the wave equation (A.1) and/or (A.5) from the covariant formulation of electromagnetism in certain stationary spacetimes see [12–14].)

The three cases can be unified as

$$\frac{\partial^2 \vec{P}}{\partial t^2} = \nabla \cdot (c^\alpha\nabla\vec{P})c^{2-\alpha} - (\nabla c^\alpha) \cdot (\nabla\vec{P})^\top c^{2-\alpha}, \tag{A.6}$$

where $\alpha = 2$ corresponds to the first two cases and $\alpha = 1$ to the third case just considered.

We are only interested in *radially-symmetric transverse* waves, so $P_r = 0$ (i.e. $E_r = B_r = 0$) and the remaining components are independent of the zenith angle ϕ. Additionally, we allow c to vary only with the radial coordinate r. Therefore, $(\nabla\vec{P})^\top$ has a zero first row while ∇c has zero second and third entries, so that $(\nabla c^\alpha) \cdot (\nabla\vec{P})^\top \equiv \vec{0}$. Now we consider the remaining term on the right-hand side of (A.6). First, we note that

$$\nabla \cdot [c^\alpha(r)\nabla\vec{P}] = \nabla c^\alpha(r) \cdot \nabla\vec{P} + c^\alpha(r)\nabla^2\vec{P}. \tag{A.7}$$

The second term above is the well-known vector Laplacian [24], while the first simplifies significantly due to our assumption that $c = c(r)$ only and $P_r = 0$. Under these assumptions, from the formulas in [25, page 507], it can be shown that

$$\nabla c^\alpha(r) \cdot \nabla\vec{P} = 0\vec{\imath}_r + \frac{\partial}{\partial r}[c^\alpha(r)]\frac{\partial P_\theta}{\partial r}\vec{\imath}_\theta + \frac{\partial}{\partial r}[c^\alpha(r)]\frac{\partial P_\phi}{\partial r}\vec{\imath}_\phi. \tag{A.8}$$

The final step in arriving at a scalar wave equation is to consider how (A.6) simplifies for the nontrivial components of $\vec{P}$. To this end, plugging (A.8) into (A.7) and realizing that the first two terms are a total derivative, we find that

$$[\nabla \cdot (c^{\alpha} \nabla \vec{P})] \cdot \vec{\imath}_\theta = \frac{\partial}{\partial r}[c^{\alpha}(r)]\frac{\partial P_\theta}{\partial r} + \frac{c^{\alpha}(r)}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial P_\theta}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_\theta}{\partial\theta}\right) - \frac{P_\theta}{\sin^2\theta}\right]$$
$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 c^{\alpha}(r)\frac{\partial P_\theta}{\partial r}\right] + c^{\alpha}(r)\left[\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_\theta}{\partial\theta}\right) - \frac{P_\theta}{r^2\sin^2\theta}\right]. \tag{A.9}$$

The expression for the $\vec{\imath}_\phi$ component turns out to be identical under these assumptions. Now we make the transformation $P_\theta(r,\theta,t) = W_\theta(r,t)/\sin\theta$. The only difficult term to convert is the second one on the right-hand side of (A.9), for which we have

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P_\theta}{\partial\theta}\right) \equiv \frac{1}{r^2}\frac{\partial^2 P_\theta}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial P_\theta}{\partial\theta}$$
$$= \left(\frac{1}{r^2\sin\theta} + \frac{2\cos^2\theta}{r^2\sin^3\theta}\right)W_\theta(r) - \frac{\cos\theta}{r^2\sin\theta}\frac{\cos\theta}{\sin^2\theta}W_\theta(r)$$
$$= \left(\frac{1}{r^2\sin\theta} + \frac{\cos^2\theta}{r^2\sin^3\theta}\right)W_\theta(r) = \frac{W_\theta(r)}{r^2\sin^3\theta}, \tag{A.10}$$

Clearly, the second and third terms on the right-hand side of (A.9) now cancel, and we obtain

$$(\text{A.6})\cdot\vec{\imath}_\theta = \frac{1}{\sin\theta}\left\{\frac{\partial^2 W_\theta}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 c^{\alpha}(r)\frac{\partial W_\theta}{\partial r}\right]\right\} = 0. \tag{A.11}$$

Again, the same holds for the $\vec{\imath}_\phi$ component and is true for both $\vec{E}$ and $\vec{B}$. Finally, the substitution $W_\theta(r,t) = r^{-1}Q_\theta(r,t)$ transforms (A.11) into

$$\frac{1}{r\sin\theta}\left\{\frac{\partial^2 Q_\theta}{\partial t^2} - c^{2-\alpha}(r)\frac{\partial}{\partial r}\left[c^{\alpha}(r)\frac{\partial Q_\theta}{\partial r}\right] + \frac{\alpha}{r}c(r)c'(r)Q_\theta\right\} = 0. \tag{A.12}$$

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