Perturbation Solution for the 2D Shallow-water Waves

C. I. Christov*, M. D. Todorov† and M. A. Christou**

*Dept of Mathematics, University of Louisiana at Lafayette, LA 70504-1010, USA
†Dept of Applied Mathematics and Computer Science, Technical University of Sofia, Bulgaria
**Dept of Mathematics, University of Nicosia, Cyprus

Abstract. The Boussinesq model of shallow water flow is considered, which contains nonlinearity and fourth-order dispersion. Boussinesq equation has been one of the main soliton models in 1D. To find its 2D solutions, a perturbation series with respect to the small parameter $\varepsilon := c^2$ is developed in the present work, where c is the phase speed of the localized wave. Within the order $O(\varepsilon^2) = O(c^4)$, a hierarchy is derived consisting of fourth-order ordinary differential equations (ODEs). The Bessel operators involved are reformulated to facilitate the creation of difference schemes for the ODEs from the hierarchy. The numerical scheme uses a special approximation for the behavioral condition in the singularity point (the origin). The results of this work show that at infinity the 2D wave shape decays algebraically, rather than exponentially as in the 1D cases. The new result can be instrumental for understanding the interaction of 2D Boussinesq solitons, and for creating more efficient numerical algorithms explicitly acknowledging the asymptotic behavior of the solution.

Keywords: Boussinesq equation, two-dimensional solitary waves, perturbation method
PACS: 44.05.+e, 44.10.+i, 05.60.-k

INTRODUCTION

Boussinesq's equation (BE) was the first model for the propagation of surface waves over shallow inviscid fluid layer [1, 2] which contained dispersion. Boussinesq found an analytical solution of his equation and thus proved that the balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. This discovery can be properly termed 'Boussinesq Paradigm' (see, *e.g.*, [3, 4] for the details).

Apart from the significance for the shallow water flows, this paradigm is very important for understanding the particle-like behavior of localized waves which behavior was discovered in the 1960ies (the so-called 'collision property'), and the localized waves were called *solitons* in [5]. The localized waves which can retain their identity during interaction appear to be a rather pertinent model for particles, especially if some mechanical properties (such as mass, energy, momentum) are conserved by the governing equations. For this reason they are called quasi-particles (QPs). In 1D, a plethora of deep mathematical results have been obtained for solitons (see [6, 7]). The success was contingent upon the existence of an analytical solution of the respective nonlinear dispersive equation.

The overwhelming majority of the analytical and numerical results obtained so far are for one spatial dimension, while in multidimension, much less is possible to achieve analytically, and almost nothing is known about the unsteady solutions that involve interactions, especially when the Boussinesq equations containg different dispersions and nonlinearities are involved. The 2D case is relatively better studied in the case of the so-called Kadomtsev-Petviashvili equation (KPE), which has fourth derivatives only in one of the spatial direction, while in the other direction, the highest-order is second. Interesting analytical results are obtained for the solutions of KPE, which are localized in the direction with the fourth-order derivative, and are periodic in the other direction (see, *e.g.*, [4, 8, 9] and the literature cited therein).

After the resound success of the QP concept in 1D, one may have expected that the attention would have turned in full swing to multidimension. Unfortunately, for the time being, the 2D Boussinesq model is still less amenable to analytical techniques. The ultimate goal is to find the collision properties of the 2D localized solutions, but even the existence of 2D stationary-propagating localized solution cannot be established numerically. This requires the development of numerical techniques. One of the main difficulties for the difference schemes lies in the inevitable reducing of the infinite interval to a finite one. This can be surmounted if a spectral method is used with a basis system of localized functions which automatically acknowledge the requirement that the solution belongs to $L^2(-\infty,\infty)$ space. Along these lines, a specialized Galerkin spectral technique was proposed in ([10]), and applied to the 2D stationary propagating 2D Boussinesq wave in [11, 12]. A perturbation technique for slow to moderate phase speed has been elaborated in [13]. A special kind of boundary conditions of the type of perfectly matched layer were used in [14] and

Application of Mathematics in Technical and Natural Sciences
AIP Conf. Proc. 1404, 49-56 (2011); doi: 10.1063/1.3659903

the results were confirmed with high accuracy by a difference scheme.

The profiles obtained for the stationary propagating QP were used for a first time for time evolution of the solution in [15], where an innovative idea (see [16] and the literature cited therein) for Reimann-like solver and transfer it to the case under consideration. The time evolution as computed in [15] showed that for phase speeds lesser than 0.3 the shape moved steadily until approximately 12 dimensionless time units and then dispersed under the influence of the higher-order derivatives. Respectively, if the phase speed exceeds the critical value 0.3, the nonlinearity dominates, but in this case the solution blows up. In order to ensure that this is not a numerical artifact, we constructed a difference scheme and verified the findings of the pioneering work with high accuracy (see [17]). This means that if the cause of blow-up is removed, we can expect to extend the interval in which the solitary wave behaves as a QP, if we are able to solve for higher phase speeds.

In 1D, it was shown in [18, 19] that the quadratic nonlinearity of the equation is the cause of the blow up. The appearance of blow up was confirmed in the numerical investigations [3]. We have embarked on providing the analytical tools for obtaining the estimates in 2D (see the work in the present volume [20]). Yet, it is important to modify the nonlinearity of the Boussinesq equations in a manner that the blow-up can be avoided. We take a clue of the work [21] where a cubic and quintic terms were shown to replace the quadratic nonlinearity in some models with significance in theory of atomic chains. Later on, it was shown numerically in [22] that for a very wide range of phase speeds, no blow up occurs. Guyed by this example we consider in the present work a Boussinesq equation with cubic and quintic nonlinear terms and use the perturbation technique to find the stationary propagating solitary wave of this model.

THE QUBIC-QUINTIC BOUSSINESQ PARADIGM EQUATION (QQBPE)

We focus here on the following two-dimensional amplitude equation:

$$w_{tt} = \Delta\left[w - \alpha(w^3 - \sigma w^5) + \beta_1 w_{tt} - \beta_2 \Delta w\right], \tag{1}$$

where w is the surface elevation, $\beta_1, \beta_2 > 0$ are two dispersion coefficients. It is easily shown that one can introduce new spatial and temporal coordinates for which the one of the dispersion coefficients can be made equal to unity. We prefer to have $\beta_2 = 1$, because we will consider only the full fledged Paradigm equations when $\beta_2 \neq 0$. The parameter σ accounts for the relative importance of the quintic nonlinearity term. We term Eq. (1) the Qubic-Quintic Boussinesq Paradigm Equation (or QQBPE). A note on the notation: in the original BE as related to the water waves, the nonlinear term has a positive sign, and the solutions are actually depressions for the subcritical case. Here we have deliberately changed the sign for the sake of the presentation.

Since the amplitude parameter α can be eliminated by rescaling the solution with simultaneous change of σ, without loosing the generality we will consider here only the case $\alpha = 1$. In order to derive the energy law for Eq. (1), we rewrite it as the following system

$$w_t = \Delta q, \qquad q_t = w - (w^3 - \sigma w^5) + \beta_1 w_{tt} - \beta_2 \Delta w. \tag{2}$$

Then the energy law reads

$$\frac{dE}{dt} = 0, \quad E = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[(\nabla q)^2 + w^2 - \frac{1}{2}w^4 + \frac{\sigma}{3}w^6 + \beta_1 w^2 + \beta_2(\nabla w)^2\right]\mathrm{d}x\mathrm{d}y. \tag{3}$$

Unlike the BPE with quadratic nonlinearity, when the amplitude increases, the quintic term in QQBPE for reasonably large σ will dominate and will make the energy functional positive, which limits the increase of the amplitude. All this means that no blow-up can be expected for QQBPE if the bi-quadratic form $1 - \frac{1}{2}w^2 + \frac{1}{3}\sigma w^4$ is always positive when $\sigma > \frac{3}{16}$. However, this case is clearly of no interest, because with the strictly positive-definite energy, the existence of bifurcation and the appearance of a nontrivial solution-like solution may not be possible. Let us concentrate our attention to the case $0 < \sigma \leq \frac{3}{16}$. Then for smaller w, the term in the energy containing only w can become even negative and the balance with the positive terms containing the gradients can define a permanent structure (solitary wave). In the above interval for σ, the bi-quadratic form adopts the form

$$1 - \tfrac{1}{2}w^2 + \tfrac{1}{3}\sigma w^4 = \tfrac{1}{3}\sigma\left[(w^2 - w_1)(w^2 - w_2)\right], \quad \text{where} \quad w_{1,2} := \tfrac{3}{4}\sigma^{-1}\left(1 \pm \sqrt{1 - \tfrac{16}{3}\sigma}\right).$$

It is clear that the last term can have negative values for $w_2 < w^2 < w_1$. This means that in the spatial intervals where this happens, it will act to destabilize the trivial solution. Yet, a blow-up cannot be reached because immediately after w^2 becomes greater than w_1 the complete positive definiteness of the energy functional is restored.

Note that the blow-up occurs when the negative term can start increasing in time. This happens when the amplitude of the function w increases.

To find a stationary moving solitary wave we introduce relative coordinates $\hat{x} = x - c_1 t,\ \hat{y} = y - c_2 t$, in a frame moving with velocity (c_1, c_2). Since there is no evolution in the moving frame $v(x,y,t) = u(\hat{x},\hat{y})$, the following equation holds for u:

$$\begin{aligned}(c_1^2 u_{\hat{x}\hat{x}} + 2c_1c_2 u_{\hat{x}\hat{y}} + c_2^2 u_{\hat{y}\hat{y}}) &= (u_{\hat{x}\hat{x}} + u_{\hat{y}\hat{y}}) - [(u^3-\sigma u^5)_{\hat{x}\hat{x}} + (u^3-\sigma u^5)_{\hat{y}\hat{y}}] - (u_{\hat{x}\hat{x}\hat{x}\hat{x}} + 2u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}}) \\ &+ \beta_1[c_1^2(u_{\hat{x}\hat{x}\hat{x}\hat{x}} + u_{\hat{x}\hat{x}\hat{y}\hat{y}}) + 2c_1c_2(u_{\hat{x}\hat{x}\hat{x}\hat{y}} + u_{\hat{x}\hat{y}\hat{y}\hat{y}}) + c_2^2(u_{\hat{x}\hat{x}\hat{y}\hat{y}} + u_{\hat{y}\hat{y}\hat{y}\hat{y}})]. \qquad (4)\end{aligned}$$

The so-called asymptotic boundary conditions (a.b.c.) read $u \to 0$, for $\hat{x} \to \pm\infty$, $\hat{y} \to \pm\infty$. The a.b.c.'s are invariant under rotation of the coordinate system, hence it is enough to consider solitary propagating wave along one of the coordinate axes, only. We chose $c_1 = 0$, $c_2 = c \neq 0$. Without fear of confusion we will 'reset' the names of the independent variables to x, y and omit in what follows the hat over the function u.

PERTURBATION METHOD

We follow [13] and create an asymptotic solution valid for small phase speeds of the soliton. We set the amplitude parameter $\alpha = 1$, because it can always be eliminated by rescaling the solution. We can also select $\beta_2 = 1$. This leaves us with only one parameter, β_1, apart from the phase speed c.

The small parameter does not multiply the highest derivative, hence the expansion is regular (see, *e.g.*, [23]). When $c = 0$, the solution possesses a radial symmetry, and we consider the expansion

$$u(x,y) = u_0(r) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + O(\varepsilon^3), \quad r = \sqrt{x^2+y^2}. \qquad (5)$$

for which we obtain the following equations

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)^3 \approx u_0^3 + 3\varepsilon u_0 u_1^2 + 3\varepsilon^2(u_0 u_1^2 + u_0^2 u_2) + O(\varepsilon^3), \qquad (6)$$

$$(u_0 + \varepsilon u_1 + \varepsilon^2 u_2)^5 \approx u_0^5 + 5\varepsilon u_0^4 u_1 + 10\varepsilon^2 u_0^3 u_1^2 + 5\varepsilon^2 u_0^4 u_2 + O(\varepsilon^3). \qquad (7)$$

Now, neglecting the terms of order $O(\varepsilon^3)$, we get for the three lowest orders in ε the following system

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\left[u_0(r) - u_0^3(r) + \sigma u_0^5(r) - \frac{1}{r}\frac{d}{dr}r\frac{du_0}{dr}\right] = 0, \qquad (8a)$$

$$\varepsilon\left[-\frac{d^2}{dy^2}u_0 + \beta_1\frac{d^2}{dy^2}\Delta u_0 + \Delta u_1 - 3\Delta(u_0^2 u_1) + 5\sigma\Delta(u_0^4 u_1) - \Delta^2 u_1\right] = 0, \qquad (8b)$$

$$\varepsilon^2\left[-\frac{d^2}{dy^2}u_1 + \beta_1\frac{d^2}{dy^2}\Delta u_1 + \Delta u_2 - 3\Delta(u_0 u_1^2 + u_0^2 u_2) + 10\sigma\Delta(u_0^3 u_1^2) + 5\sigma\Delta(u_0^4 u_2) - 2\Delta(u_0 u_2) - \Delta^2 u_2\right] = 0. \qquad (8c)$$

We prefer to treat the above system in polar coordinates because then the region is unbounded only with respect to one of the variables (the polar radius r). The connection between Cartesian and polar coordinates is given by $x = r\cos(\theta), y = r\sin(\theta)$, where θ is the polar angle. Now for the derivative with respect to y we have

$$\frac{\partial^2}{\partial y^2} := \sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\sin 2\theta}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\sin 2\theta}{r^2}\frac{\partial}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r}. \qquad (9)$$

Applying twice this operator we find the expression for the fourth derivative with respect to y. The Laplace operator in polar coordinates is well known and is omitted here.

Denote $F(r) = u_0(r)$. When manipulating the equation for u_1, we observe that the operator in Eq. (9) has to be applied only to the function u_0 which is independent of the polar angle θ and we get the following

$$\frac{\partial^2}{\partial y^2}u_0(r) = \sin^2\theta\frac{d^2F}{dr^2} + \frac{\cos^2\theta}{r}\frac{dF}{dr} = \frac{1}{2}\left(\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr}\right) + \frac{1}{2}\left(-\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr}\right)\cos(2\theta),$$

$$\frac{\partial^2}{\partial y^2}\Delta u_0(r) = \sin^2\theta\frac{d^2P}{dr^2} + \frac{\cos^2\theta}{r}\frac{dP}{dr} = \frac{1}{2}\left(\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr}\right) + \frac{1}{2}\left(-\frac{d^2P}{dr^2} + \frac{1}{r}\frac{dP}{dr}\right)\cos(2\theta), \quad P(r) := \frac{1}{r}\frac{d}{dt}r\frac{1}{r}\frac{d}{dt}F(r).$$

In this work we limit ourselves to the $O(\varepsilon)$ approximation, because it is usually enough for the purposes of an initial condition, and the higher order approximations become unwiledy for $\beta_1 \neq 0$. Then we can recast Eq. (8b), as follows

$$\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left[u_1(r,\theta) - 3F^2(r)u_1(r,\theta) + 5F^4(r)u_1(r,\theta) - \left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u_1(r,\theta)\right]$$
$$= \sin^2\theta\frac{d^2}{dr^2}F(r) + \frac{\cos^2\theta}{r}\frac{d}{dr}F(r) - \beta_1\left[\sin^2\theta\frac{d^2}{dr^2}P(r) + \frac{\cos^2\theta}{r}\frac{d}{dr}P(r)\right]. \quad (10)$$

The form of the right-hand side of Eq. (10) suggests that the following type of solution can be found

$$u_1(r,\theta) = G(r) - \beta_1 Q(r) + [H(r) - \beta_1 R(r)]\cos(2\theta). \quad (11)$$

THE GOVERNING SYSTEM

Before proceeding with the derivation of the system for the functions F,G,H and P,Q,R we observe that the higher-order Bessel operators involved in those equations have the form (see [13]):

$$\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{4}{r^2} \equiv r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2, \quad \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{16}{r^2} \equiv r^3\frac{d}{dr}\frac{1}{r^7}\frac{d}{dr}r^4. \quad (12)$$

We can integrate Eq. (8a) and set the two integration constants equal to zero. Thus we get a lower-order equation for $F(r)$:

$$F(r) - F^3(r) + \sigma F^5(r) - \frac{1}{r}\frac{d}{dr}r\frac{dF}{dr} = 0. \quad (13a)$$

Upon substituting Eq. (11) into Eq. (10), and integrating twice under the asymptotic boundary conditions, we get the following equations for G and Q:

$$-G(r) + 3F^2(r)G(r) - 5\sigma F^4(r)G(r) + \frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} = -\frac{1}{2}F(r), \quad (13b)$$
$$-Q(r) + 3F^2(r)Q(r) - 5\sigma F^4(r)Q(r) + \frac{d^2Q}{dr^2} + \frac{1}{r}\frac{dQ}{dr} = -\frac{1}{2}P(r). \quad (13c)$$

Implementing consistently the above idea (grouping the terms with the same dependence on θ) we get the equations for the other functions, H,R, namely:

$$r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2\left[-H(r) + 3F^2(r)H(r) - 5\sigma F^4(r)H(r) + r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2H(r)\right] = \frac{1}{2}\left[\frac{d^2}{dr^2}F(r) - \frac{1}{r}\frac{d}{dr}F(r)\right], \quad (13d)$$
$$r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2\left[-R(r) + 3F^2(r)R(r) - 5\sigma F^4(r)R(r) + r\frac{d}{dr}\frac{1}{r^3}\frac{d}{dr}r^2R(r)\right] = \frac{1}{2}\left[\frac{d^2}{dr^2}P(r) - \frac{1}{r}\frac{d}{dr}P(r)\right]. \quad (13e)$$

When one is faced with singularities that arise from the use of specific coordinates (*e.g.*, polar coordinates), he has to ensure the proper behavior of the functions in the point of singularity by imposing additional (purely mathematical) conditions in the geometric singularity called 'behavioral' (see [24]). The behavioral conditions at the origin arise from the fact that there is a singularity in the operator:

$$G'(0) = G'''(0) = Q'(0) = Q'''(0) = H'(0) = H'''(0) = R'(0) = R'''(0) = 0, \quad (14a)$$

while the behavioral conditions at infinity (called asymptotic boundary conditions or a.b.c.) are given by

$$G(r), Q(r), H(r), R(r) \to 0 \quad \text{for} \quad r \to \infty. \quad (14b)$$

The equations possess non-trivial solutions provided a nontrivial solution $F(r) \not\equiv 0$ is found, because the rest of the equations are linear. Thus, one can tackle the bifurcation problem while finding the function $F(r)$.

RESULTS

For the very elaborate technical details of the difference schemes for the above equations we refer the reader to [13]. Respectively, the details of the application of the spectral methods can be found for the 1D case in [25] and for 2D case in [26, 11]. For the lack of space we do not present here the verifications of the present results by means of the spectral algorithm from [11]. In Figure 1 we present the results for the sought three functions. The right panel shows the result for F, G, H in linear, log-linear and log-log scale alongside with the comparison to the best-fit approximations. It is clearly seen that the asymptotic decay for H is indeed proportional to the inverse square of the polar coordinate. The results also testify that the 'computational infinity' $r_\infty = 1000$ is fully adequate for finding $H(r)$ with good accuracy. Note that because of the exponential decay of F and G we augment the needed values of their grid versions with zeros in the region $50 < r < 1000$. Respectively, Figure 2 gives the result for the functions corresponding to terms that are

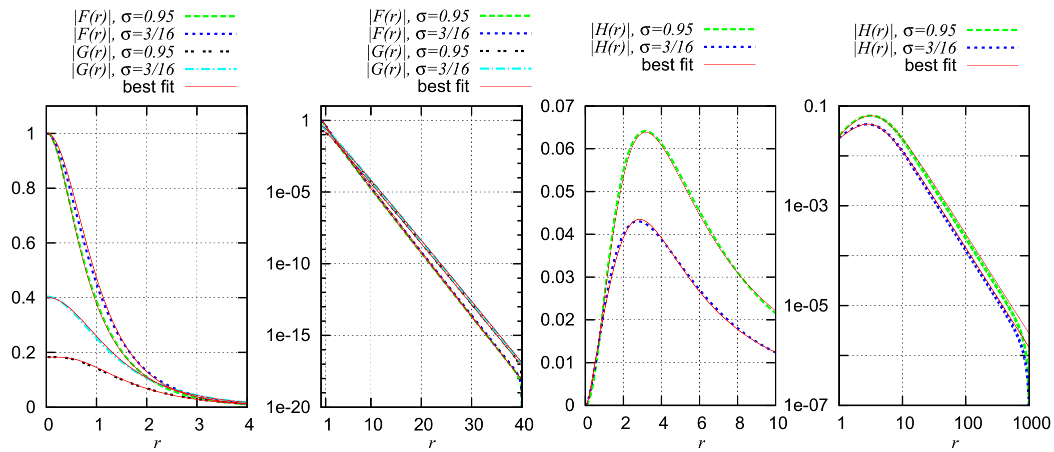


FIGURE 1. Finite-difference solutions for the $F(r), G(r), H(r)$ for two values of $\sigma = 0.95$ and $\sigma = 3/16$. The left panels present the behavior near the origin and the right panels present the behavior in the far fields in logarithmic coordinates. All panels give the best-fit functions denoted further as $f(r)$, $g(r)$ and $h(r)$ whose analytical representations shown in the Table 1, rows 1, 2 and 3

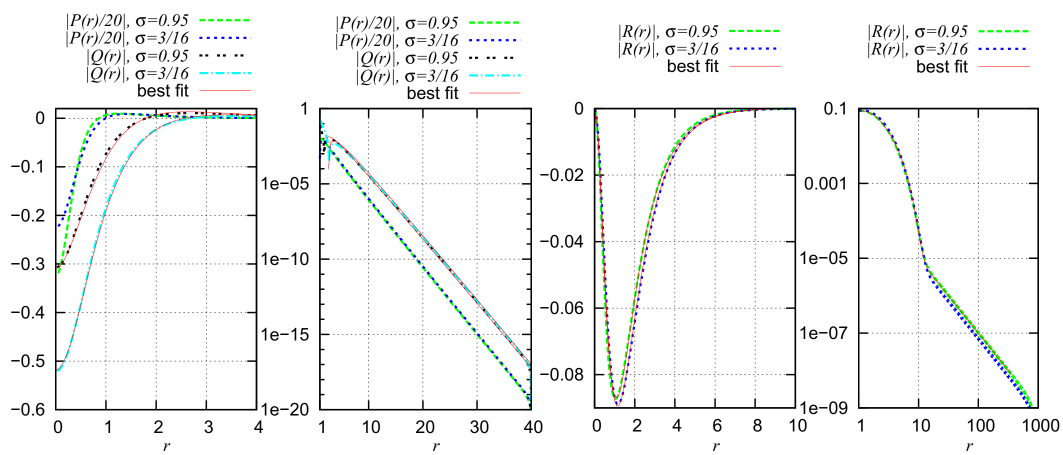


FIGURE 2. Finite-difference solutions for the $P(r), Q(r), R(r)$ for two values of $\sigma = 0.95$ and $\sigma = 3/16$. The left panels present the behavior near the origin and the right panels present the behavior in the far fields in logarithmic coordinates. All panels give the corresponding best-fit functions $\hat{f}(r)$, $\hat{g}(r)$ and $\hat{h}(r)$, whose analytical expressions are presented in the Table 1, rows 4, 5 and 6

multiplied by β_1.

Since, the primary purpose of this short note is to create an initial condition for time stepping algorithms, we need to present the results, as best-fit analytic approximations for the functions F, G, H. Essentially, this is the same approach as in [13] but now we are more restricted because the new time stepping algorithm developed in [17] requires also initial conditions for the Laplacian of the functions. This prevents using first powers in the approximation. We found the following best-fit approximations for the shape of the stationary propagating solitons for $\beta_2 = 1$ and two concrete values of the weight coefficient σ (see Table 1), namely:

$$w^s(x,y,t;c) = f(x,y) + c^2[g(x,y) + h(x,y)\cos(2\theta)] - \beta_1 c^2[\hat{g}(x,y) + \hat{h}(x,y)\cos(2\theta)], \tag{15}$$

where $\theta(x,y) = \arctan(y/x)$.

TABLE 1. Best-fit functions depending on the weight coefficient σ

σ	0.95	3/16
$f(x,y)$	$1.0032\,\frac{1+0.32r^2}{(1+1.86r^2+0.032r^4)^{0.75}}\,\mathrm{sech}(r)$	$1.0032\,\frac{1+0.31r^2}{(1+0.55r^2)^{1.25}}\,\mathrm{sech}(r)$
$g(x,y)$	$0.203(1+0.1r^2)^{0.25}\,[1.2\mathrm{sech}(r) - 0.3\mathrm{sech}(2r)]$	$0.444\,(1+0.04r^2)^{0.25}\,\mathrm{sech}(r)$
$h(x,y)$	$\frac{0.05r^2+0.071r^4}{1+3.2r^2+0.6r^4+0.026r^6}$	$0.2\,\frac{0.064r^2+0.0628r^4}{1+4r^2+0.75r^4+0.04r^6}$
$\hat{g}(x,y)$	$(1+3r^2)^{0.25}\,[0.085\mathrm{sech}(r) - 0.44\mathrm{sech}(1.86r)$	$(1+3r^2)^{0.25}$
	$+0.102\mathrm{sech}(2.2r) - 0.056\mathrm{sech}(4r)]$	$\times\,[0.167\mathrm{sech}(r) - 0.295\mathrm{sech}(1.3r) - 0.39\mathrm{sech}(2.2r)]$
$\hat{h}(x,y)$	$\frac{-0.36r^2}{(1+r^2)^2}\,[0.87\mathrm{sech}(2r) + 0.02r^2\mathrm{sech}(1.51r) - 0.001]$	$\frac{-0.33r^2}{(1+r^2)^2}\,[0.87\mathrm{sech}(1.89r) + 0.02r^2\mathrm{sech}(1.49r) - 0.00078]$

We need to make sure here that the Laplacians of our functions F, G, and H are also smooth and well behaved, in order to be able to use them in algorithms of the type of [17]. In Figure 3 the Laplacians of the three main functions F, G, H are plotted. The results for Q and R are very similar.

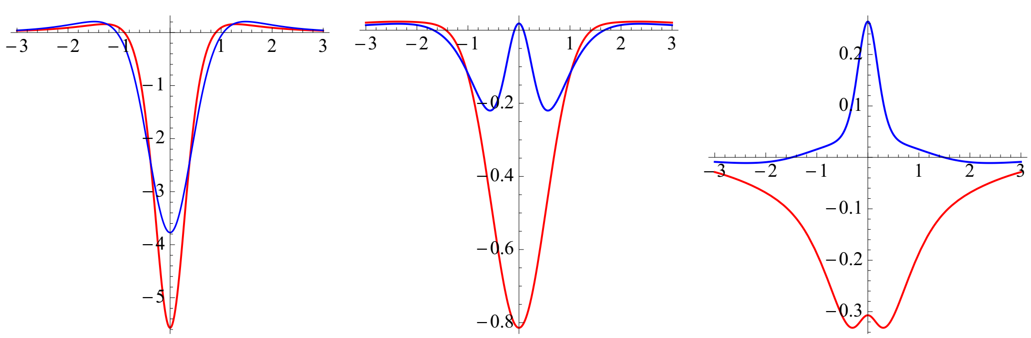


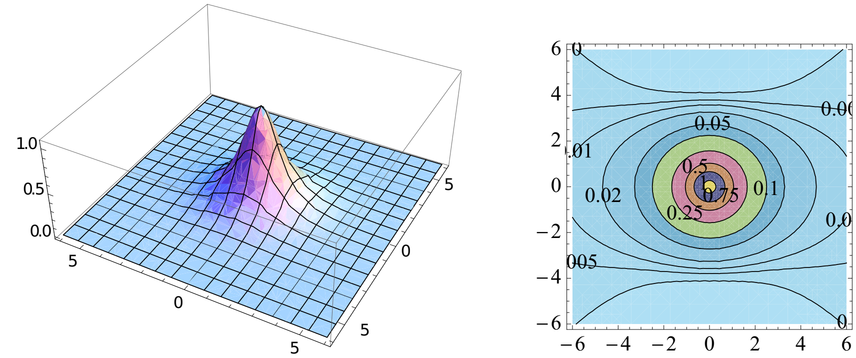
FIGURE 3. Cross sections at $y = 0$ of the Laplacians $\Delta F(r)$ (left panel), $\Delta G(r)$ (middle panel), and $\Delta H(r)$ (right panel) for the two selected values of σ: 0.95 (red); 3/16 (blue)

In the end, we present in Figure 4 the shape of the solitary wave and the contour lines of the cross-sections for four different combination of parameters.

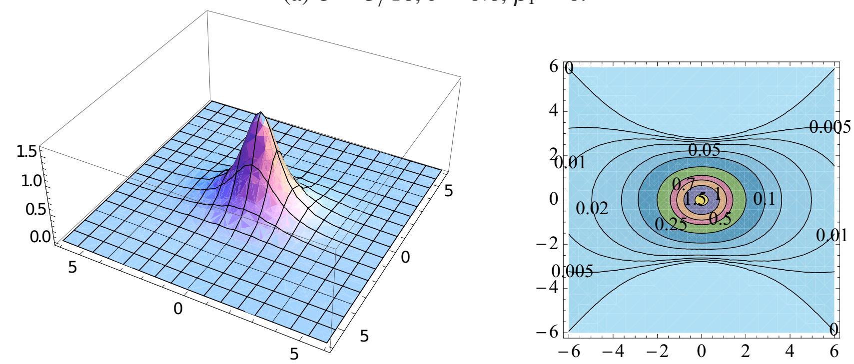
It is observed that for nontrivial value of the phase speed (specifically $c = 0.6$) the shape undergoes significant contraction in the direction of the propagation. In order to demonstrate this we use a special set of non-uniformly distributed values for the contour lines. In addition, when β_1 is increased from 0 to 3, the contraction becomes even more pronounced. This is similar to the case of BE with quadratic nonlinearity [13], in the present case, no depressions are observed before and after the main shape. This means, that the selected here nonlinearity is more adequate in physical sense, and is better suited to the purpose of investigating permanent waves.

CONCLUSION

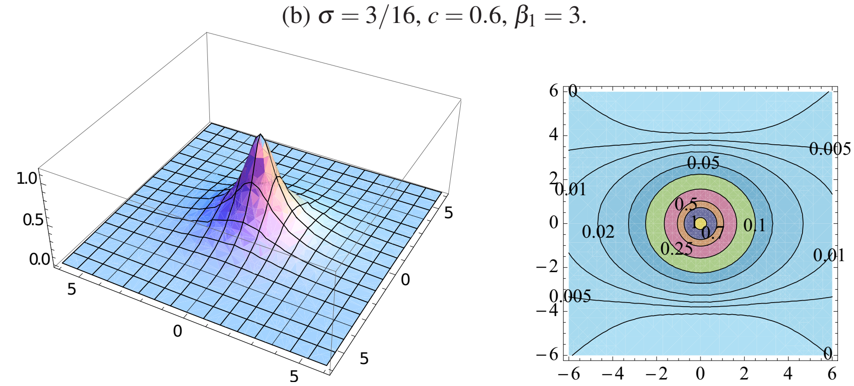
In this present paper we apply to the case of Boussinesq equation with cubic and quintic nonlinearity a perturbation technique based on the asymptotic expansion for small phase speed, c, and carry out the solution including terms up to $O(c^2)$. Within the adopted asymptotic order, we reduce the original 2D problem to three fourth-order ODEs for



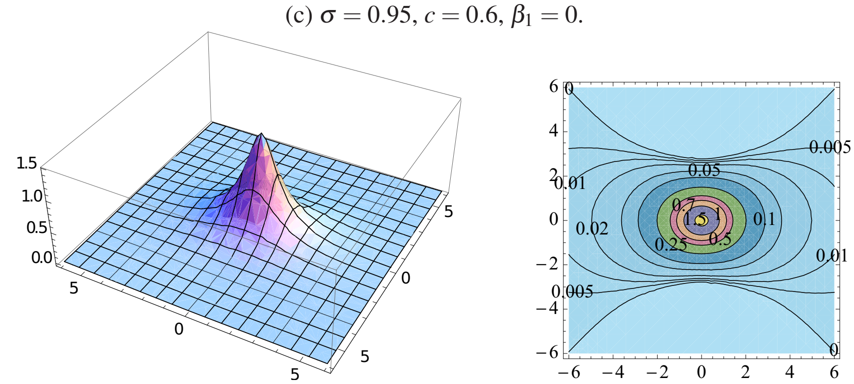
(a) $\sigma = 3/16$, $c = 0.6$, $\beta_1 = 0$.



(b) $\sigma = 3/16$, $c = 0.6$, $\beta_1 = 3$.



(c) $\sigma = 0.95$, $c = 0.6$, $\beta_1 = 0$.



(d) $\sigma = 0.95$, $c = 0.6$, $\beta_1 = 3$.

FIGURE 4. Results for different parameters. Left panel: surface plot. Right panel: contours.

functions that depend only on the radial variable. Following our previous work we construct special approximations on staggered grids which satisfy automatically the behavioral boundary conditions.

Our results confirm that even for this more intricate nonlinearity, the asymptotic decay of the wave profile is algebraic rather than exponential (which is the case with the axisymmetric profile of the standing 2D soliton). This means that the profile of the 2D soliton is not robust: even the presence of a very small phase speed, makes the behavior of the shape in the far field qualitatively different.

Similarly to the Boussinesq equation with quadratic nonlinearity (investigated in previous works of CIC), the profile is contracted relatively in the direction of motion and the contraction increases with the increase of the phase speed. The role of the dispersion parameter β_1 is shown to enhance the contraction. For larger β_1 the contraction is more pronounced. The most important difference from case with quadratic nonlinearity is that no depressions (negative hight of the profile) are formed in the front and back of the main hum.

The main utility of the solution obtained here is as a precise initial condition for time stepping algorithms that may be used to investigate whether the two dimensional shapes can be permanent, *i.e.*, Quasi-Particles.

ACKNOWLEDGMENTS

This work is supported by the National Science Fund of Ministry of Education, Science, and Youth of Republic Bulgaria under grant DDVU02/71.

REFERENCES

1. J. V. Boussinesq (1871) *Comp. Rend. Hebd. des Seances de l'Acad. des Sci.* **73**, 256–260.
2. J. V. Boussinesq (1872) *Journal de Mathématiques Pures et Appliquées* **17**, 55–108.
3. C. I. Christov, and M. G. Velarde (1994) *J. Bifurcation & Chaos* **4**, 1095–1112.
4. C. I. Christov, G. A. Maugin, and A. Porubov (2007) *C.R. Mecanique* **335**, 521–535, doi:10.1016/j.crme.2007.08.006.
5. N. J. Zabusky and M. D. Kruskal (1965) *Phys. Rev. Lett.* **15**, 240–243.
6. M. J. Ablowitz, and H. Sigur, *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia, 1981.
7. A. C. Newell, *Solitons in Mathematics and Physics*, SIAM, Philadelphia, 1985.
8. A. V. Porubov, G. A. Maugin, and V. V. Mateev (2004) *Nonlinear Mechanics* **39**, 1359–1370.
9. A. V. Porubov, F. Pastrone, and G. A. Maugin (2004) *C. R. Mecanique* **332**, 513–518.
10. C. I. Christov (1982) *SIAM J. Appl. Math.* **42**, 1337–1344.
11. M. A. Christou and C. I. Christov (2007) *Math. Comput. Simul.* **74**, 82–92.
12. M. A. Christou and C. I. Christov, "Galerkin Spectral Method for the 2D Solitary Waves of Boussinesq Paradigm Equation," in *AMiTaNS'09*, edited by M. D. Todorov, and C. I. Christov, AIP CP **1186**, American Institute of Physics, Melville, NY, 2009, pp. 217–225.
13. C. I. Christov and J. Choudhury (2011) *Mech. Res. Commun.*, galley proofs.
14. C. I. Christov, *Math. Comp. Simul.* (Appeared online August 10, 2010), `http://dx.doi.org/10.1016/j.matcom.2010.07.030`.
15. A. Chertock, C. I. Christov, and A. Kurganov, "Central-Upwind Schemes for the Boussinesq Paradigm Equation," in *Computational Sci., & High performance Computing IV, NNFM*, edited by E. Krause *et al*, 2011, pp. 267–281.
16. A. Kurganov and E. Tadmor (2002) *Numer. Meth. Part. Diff. Eq.* **18**, 584–608.
17. C. I. Christov, N. Kolkovska, and D. Vasileva, "On the Numerical Simulation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation," in *NMA 2010*, edited by I. Dimov, S. Dimova, and N. Kolkovska, 2011, vol. **6046** of *LNCS*, pp. 386–394.
18. S. K. Turitzyn (1993) *Phys. Rev. E* **47**, R13–R16.
19. S. K. Turitsyn (1993) *Phys. Rev. E* **47**, R769–R799.
20. N. Kutev, N. Kolkovska, M. Dimova, and C. I. Christov, "Theoretical and numerical aspects for global existence and blow up for the solutions to Boussinesq Paradigm Equation," in *AMiTaNS'11*, edited by M. D. Todorov, and C. I. Christov, American Institute of Physics, Melville, NY, 2011, accepted in this volume.
21. G. A. Maugin and S. Cadet (1991) *Int. J. Engng. Sci.* **29**, 243–258.
22. C. I. Christov and G. A. Maugin, "Numerics of Some Generalized Models of Lattice Dynamics," in *Nonlinear Waves in Solids, (ASME Book N0 AMR137)*, edited by J. Wegner, and R. Norwood, 1995, pp. 374–379.
23. J. D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell Pub. Co, Waltham, 1968.
24. J. P. Boyd, *Chebishev and Fourier Spectral Methods*, Dover, New York, 2001, 2nd edn.
25. C. I. Christov and K. L. Bekyarov (1990) *SIAM J. Sci. Stat. Comp.* **11**, 631–647.
26. C. I. Christov (1995) *Annuaire de l'Univ. Sofia, Livre 2 – Mathématiques Appliqee et Informatique* **89**, 169–179.