Theoretical and Numerical Aspects for Global Existence and Blow Up for the Solutions to Boussinesq Paradigm Equation

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Abstract. In this paper we prove that the global existence and the blow up of the weak solutions to Boussinesq Paradigm Equation (BPE) depend not only on the initial energy but also on the profiles of the initial data.

The constant d of the critical initial energy which guaranties the above properties of the solution is found explicitly by means of the exact constant of the Sobolev embedding theorem. We demonstrate numerically in the one dimensional case that this constant d is the best possible one for the global existence and a lack of global existence.

In this way we can find the intervals for the velocities c of the solitary wave solutions to BPE in which the solution to BPE with initial data close to the solitons exists globally in time. Thus for different parameters of BPE we give numerically some ranges of the stability of the solitary waves.

Keywords: Boussinesq Paradigm Equation, global existence, blow up, solitary waves, stability theory
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INTRODUCTION

Boussinesq [1, 2] introduced the generalized wave equation containing dispersion and was able to explain the appearance of the "Permanent Wave" that can propagate at long distances without changing its shape.

In numerical experiments it was observed that the solution blows-up in finite time, but the first analytical explanation of this fact was provided by Turitsyn [17, 18] who made use of a technique proposed in [10] and proved that the amplitude of the solution can blow up in finite time, while its supports shrinks to zero simultaneously. Alternatively, the effect can be found in the literature under the name "collapse" of the solution of Boussinesq equation. The effect is due to the non-positive definiteness of the energy functional, and is actually an artifact of the simplifications used to derive the amplitude equations for the surface from the full system of equations for the flow in inviscid shallow layer (see the discussion in [6]).

Using an energy-conserving scheme the blow up for the Boussinesq Equation (BE) and Regularized Long Wave Equation (RLWE) was thoroughly investigated numerically in [9, 5, 6, 7] and it was shown that it inevitably takes place when the absolute value of the initial energy of the system is large enough. For an useful survey of the results we refer the reader to another contribution in the present Proceedings [8].

POSING THE PROBLEM

We consider the Boussinesq Paradigm Equation (BPE)

$$\begin{aligned} u_{tt} - \Delta u - \beta_1 \Delta u_{tt} + \beta_2 \Delta^2 u = \Delta f(u) \qquad &\text{for} \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x) \qquad &\text{for} \qquad x \in \mathbb{R}^n \end{aligned} \tag{1}$$

with nonlinear term $f(u) = \alpha |u|^p$, where $\alpha > 0$, $\beta_1 \geq 0$, $\beta_2 > 0$ are real constants, $1 < p < \infty$ for $n = 1,2$ and $(n+2)/n \leq p \leq (n+2)/(n-2)$ for $n \geq 3$. The initial data are in the Banach spaces

$$u_0 \in \mathrm{H} = \{u \in \mathrm{H}^1; (-\Delta)^{-1/2} u \in \mathrm{L}^2\}, \qquad u_1 \in \mathrm{L} = \{u \in \mathrm{L}^2; (-\Delta)^{-1/2} u \in \mathrm{L}^2\}$$

with the norms

$$\|u\|^2_{\mathrm{L}} = \|u\|^2_{\mathrm{L}^2} + \|(-\Delta)^{-1/2} u\|^2_{\mathrm{L}^2}, \qquad \|u\|^2_{\mathrm{H}} = \|u\|^2_{\mathrm{H}^1} + \|(-\Delta)^{-1/2} u\|^2_{\mathrm{L}^2},$$

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where $(-\Delta)^{-s}u = F^{-1}\left(|\xi|^{-2s}F(u)\right)$ for $s>0$ and $F(u)$, $F^{-1}(u)$ are the Fourier transformation and the inverse Fourier transformation, respectively. Weak solution of (1) in $[0,T)\times\mathbb{R}^n$ is a function u, $u\in \mathrm{L}^\infty([0,T);\mathrm{H}^1)$, $u_t\in \mathrm{L}^\infty([0,T);\mathrm{L})$, satisfying the identity

$$\left((-\Delta)^{-1/2}u_t,(-\Delta)^{-1/2}v\right)_{\mathrm{L}^2} + (u_t,v)_{\mathrm{L}^2} + \int_0^t \left[(u,v)_{\mathrm{L}^2} + (\nabla u,\nabla v)_{\mathrm{L}^2} + (f(u),v)_{\mathrm{L}^2}\right] d\tau = \left((-\Delta)^{-1/2}u_1,(-\Delta)^{-1/2}v\right)_{\mathrm{L}^2} + (u_1,v)_{\mathrm{L}^2}$$

for every $v\in \mathrm{H}$ and every $t\in[0,T)$.

The aim of this paper is to study when the weak solutions of (1) are globally defined or blow up for a finite time. In view of the results in [20] we find explicitly the best constant d of the initial energy introduced in (3), which is crucial for the validity of theorems in [20]. As a consequence of the knowledge of the exact constant d, we get ranges of the stability for the solitary waves of BPE depending on their velocities.

The paper is organized in the following way. In Section 3 the theorems from [20] are reformulated for general BPE while in Section 4 the main results of the paper are proved. Section 5 deals with some applications to the stability of solitary waves for BPE.

PRELIMINARIES

Let us mention some theoretical results about global existence and finite blow up of the solutions of (1). In some recent papers the Boussinesq equation was investigated in the one dimensional case in [17, 18, 13, 19, 21] and in the multidimensional case in [16, 20] for nonlinear terms $f(u)=\pm\alpha|u|^p$ or $f(u)=\alpha|u|^{p-1}u$, $p>1$. The behavior of the solutions for sign preserving or sign changing nonlinearities is quite different. Moreover, in the above articles only the special choice of the constant $\beta_2=1$ and $\beta_1=1$ or $\beta_1=0$ was considered.

In order to reformulate the results for general equation (1) as in the case when $\beta_1=\beta_2=1$ (see [16, 19, 20]) let us change the variables $\bar{x}=x/\sqrt{\beta_2}$, $\bar{t}=t$. Using the same notations equation (1) can be rewritten in the following form

$$\begin{gathered} \beta_2 u_{tt} - \Delta u - \beta_1\Delta u_{tt} + \Delta^2 u = \Delta f(u), \\ u(x,0)=u_0(\sqrt{\beta_2}x), \quad u_t(x,0)=u_1(\sqrt{\beta_2}x). \end{gathered} \tag{2}$$

Now the conservation low of the full energy of the weak solutions of (2) is given by

$$E(t)=E(u(\cdot,t),u_t(\cdot,t))=\frac{1}{2}\left[\beta_2\left\|(-\Delta)^{-1/2}u_t\right\|^2_{\mathrm{L}^2}+\beta_1\|u_t\|^2_{\mathrm{L}^2}+\|u\|^2_{\mathrm{H}^1}\right]+\frac{\alpha}{p+1}\iint_{\mathbb{R}^n}|u|^p u\,dx \equiv E(0).$$

Let us introduce the functional $J(u)$

$$J(u)=\frac{1}{2}\|u\|^2_{\mathrm{H}^1}+\frac{\alpha}{p+1}\iint_{\mathbb{R}^n}|u|^p u\,dx$$

which is a part of the full energy E and the sign preserving energy functional $I(u)$

$$I(u)=\|u\|^2_{\mathrm{H}^1}+\alpha\iint_{\mathbb{R}^n}|u|^p u\,dx.$$

By means of these functionals the critical energy constant d is defined in [20]

$$d=\inf_{u\in \mathrm{N}} J(u), \qquad \mathrm{N}=\{u\in \mathrm{H}^1; I(u)=0, \quad \|u\|_{\mathrm{H}^1}\neq 0\}. \tag{3}$$

The following Theorem 1 is an extension of Theorem 4.10 and Corollary 4.9 in [20] for general equation (2).

Theorem 1. *([20]) If $E(0)<0$ then every weak solution of (2) blows up for a finite time. If $E(0)=0$ then every weak solution of (2), except the trivial one, blows up for a finite time. When $0<E(0)<d$ then:*

(i) if $I(u_0)<0$ then the weak solution of (2) blows up for a finite time;

(ii) if $I(u_0) > 0$ then the weak solution of (2) is globally defined for $t \in [0,\infty)$.

A lower bound for the constant d is found in the same paper [20]

$$d \geq \frac{p-1}{2(p+1)} \left[\alpha C^{p+1}\right]^{-\frac{2(p+1)}{p-1}}, \quad \text{where} \quad C = C(p) = \sup_{\substack{u\in \mathrm{H}^1\\ u\neq 0}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}}.$$

In fact, in an earlier paper of Y. Liu [14] for the special equation (2) with $n=1$, $\beta_1 = 0$, $\beta_2 = 1$ and $f(u) = -|u|^{p-1}u$ the following result for d is proven

$$d = E(\varphi) = \frac{1}{2}\|\varphi\|^2_{\mathrm{H}^1} - \frac{1}{p+1}\|\varphi\|^{p+1}_{\mathrm{L}^{p+1}},$$

where φ is the positive radial $\mathrm{H}^1(\mathbb{R}^n)$ solution of the ground state equation

$$-\Delta\varphi + \varphi - |\varphi|^{p-1}\varphi = 0 \qquad \text{in} \quad \mathbb{R}^n,$$

$1 < p < \infty$ for $n = 1,2$ and $1 < p < (n+2)/(n-2)$ for $n \geq 3$. The ground state equation is solvable explicitly only for $n = 1$. In [14] the following estimate for d is given

$$\frac{(p-1)^{\frac{1}{2}}}{2(p+1)}\left[(2(p+1))^{-\frac{1}{2}}\,(p+3)^{\frac{p+3}{4(p+1)}}\right]^{-\frac{2(p+1)}{p-1}} < d < \frac{(p-1)^{\frac{1}{2}}}{2(p+1)}\left[(2(p+1))^{-\frac{1}{2}}\,(p+3)^{\frac{p+3}{4(p+1)}}\,e^{-\frac{p-1}{2(p+1)}}\right]^{-\frac{2(p+1)}{p-1}}.$$

Clearly, the value of d is crucial for the numerical calculations in proving global existence or finite blow up, as well as the stability of the solitons. Unfortunately, in the previous results only an upper and a lower bound for d are found.

MAIN RESULT

In the following theorem we get explicitly the value of d.

Theorem 2. *For d defined in (3) the equality*

$$d = \frac{p-1}{2(p+1)}\left[\alpha C^{p+1}\right]^{-\frac{2(p+1)}{p-1}} \qquad \textit{holds, where} \qquad C = C(p) = \sup_{\substack{u\in \mathrm{H}^1\\ u\neq 0}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}}.$$

Moreover, for $n = 1$ it follows that

$$d = \frac{1}{p+3}\left[\frac{2(p+1)}{\alpha}\right]^{\frac{2}{p-1}} \Gamma\left(\frac{2}{p-1}\right)^2 \Big/ \Gamma\left(\frac{4}{p-1}\right).$$

Corollary 1. *For $n = 1$, $p = 2$,* i.e.*, for equation*

$$\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = \alpha\Delta u^2,$$

we have $d = \dfrac{6}{5\alpha^2}$. Moreover, if $\alpha = 3$ then $d = \dfrac{2}{15} = 0.133\ldots$.

Idea of the proof. In the paper of E. Lieb [12] the author finds out the best constant for the embedding of H^1 in L^{p+1} for $n = 1$, $p > 1$, i.e.

$$C = C(p) = \sup_{\substack{u\in \mathrm{H}^1\\ u\neq 0}} \frac{\|u\|_{\mathrm{L}^{p+1}}}{\|u\|_{\mathrm{H}^1}} = \left\{\frac{1}{2(p+1)}\left[(p-1)(p+3)\frac{\Gamma(\frac{4}{p-1})}{\Gamma(\frac{2}{p-1})^2}\right]^{\frac{p-1}{p+1}}\right\}^{\frac{1}{2}}.$$

E. Lieb [12] proves that the equality is attained if and only if u is one of the extremal functions $Aw = A\left(\cosh(\frac{p-1}{2})x\right)^{-\frac{2}{p-1}}$, $\quad A = \text{const} \neq 0$. For the special choice of the constant $A = A_*$

$$A_* = -\left[\frac{\|w\|^2_{\mathrm{H}^1}}{\alpha\iint_{\mathbb{R}^n} w^{p+1}\,dx}\right]^{\frac{1}{p-1}}$$

the function $u_* = A_* w$ is also an extremal function for the functional $J(u)$, *i.e.*,

$$I(u_*) = 0, \quad \text{and} \quad \inf_{u\in \mathrm{N}} J(u) = J(u_*) = d. \qquad \square$$

It is important one to show whether the constant d in Theorem 2 is sharp for the validity of the statement in Theorem 1 or there exists a larger constant $D > d$ such that for $E(0) < D$ Theorem 1 still holds. In the numerical experiments to follow we demonstrate that the constant d found in Theorem 2 is the best possible one.

For this purpose we consider the equation (2) for $n = 1$, $p = 2$, $\alpha = 3$, $f(u) = 3u^2$ and pose the following initial boundary value problem

$$\begin{aligned} &\beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} = 3(u^2)_{xx}, \\ &u(x,0) = u_0(x) = -A\cosh^{-2}(\tfrac{x}{2}), \qquad u_t(x,0) = u_1(x) = \varepsilon\tilde{u}_0(x). \end{aligned} \tag{4}$$

Here the constants A and ε will be chosen sufficiently close to 0.5 and 0 respectively. The even function $\tilde{u}_0(x)$ is defined as a $\mathrm{C}^2(\mathbb{R})$ function in the following way: $\tilde{u}_0(x) = u_0(x)$ for $|x| \leq L_1$, $\tilde{u}_0(x) > 0$ for $L_1 + 1 < |x| < L_2$ and $\tilde{u}_0(x) \equiv 0$ for $|x| \geq L_2$. The positive constants $1 \ll L_1 + 1 < L_2$ are fixed such that $\int_{-\infty}^{\infty} \tilde{u}_0(x)\,dx \approx 0$. This means that $\tilde{u}_0(x) = \frac{\partial}{\partial x} \int_{-\infty}^{x} \tilde{u}_0(y)\,dy$ and $\tilde{u}_0(x)$ is a derivative of a $\mathrm{L}^2(\mathbb{R})$ function. So the kinetic energy of the initial velocity u_1, *i.e.*,

$$\frac{1}{2}\left[\beta_2 \left\|(-\Delta)^{-1/2}u_1\right\|^2_{\mathrm{L}^2} + \beta_1\|u_1\|^2_{\mathrm{L}^2}\right] = \frac{\varepsilon^2}{2}\left[\beta_2 \left\|\int_{-\infty}^{x} \tilde{u}_0(y)\,dy\right\|^2_{\mathrm{L}^2} + \beta_1\|\tilde{u}_0(x)\|^2_{\mathrm{L}^2}\right],$$

is finite and $u_1 \in \mathrm{L}$. Moreover, this energy is sufficiently small for all ε small enough.

To solve numerically the problem (4) we use a conservative, implicit with respect to the nonlinearities, finite difference scheme. The first conservative scheme for (4) is proposed, studied and thoroughly verified in [9] where a lot of results about collisions of different solitary waves are presented (including the path to blow-up phenomenon). The most important advantage of this scheme is that it employs internal iterations which allows one to have a conservation of the physical energy virtually up to the *round-off* error of the computer arithmetic. The concept of scheme of [9] was later on applied to various different problems and its excellent performance was crucial for obtaining results for very long times. If an ordinary scheme without internal iterations is used, the discrete energy equation contains an additional source term proportional to the time step, which can bring into view some accumulation effects that can actually precipitate an earlier blow-up or even suppression of the blow-up. A slightly different scheme and a discrete energy functional are considered. Similarly to [9], it is proved that this discrete functional is conserved in time. Thus one can conclude that we have selected the most pertinent mathematical tool to investigate numerically the blow-up. In addition, we use some specific mesh refinement in the present paper.

Numerical tests are conducted for $\beta_1 = 3/2$, $\beta_2 = 1/2$ and for different values of the constant A. Note that for $\alpha = 3$ the critical value of the initial energy d found in Theorem 2 is $d = \frac{2}{15} \approx 0.1333\ldots$.

TABLE 1. Initial energy $E(0)$ as a function of ε and the existence time of the solutions of (4); $x \in [-50, 50]$, $0 \leq t \leq 50$, t^* - blow up time; (a): $A = 0.495$, $I(0) \approx 0.007841 > 0$; (b): $A = 0.499$, $I(0) \approx 0.001594 > 0$

ε	$E(0)$	existence time
0.0100	0.136705	$t^* \approx 12.25$
0.0045	0.133981	$t^* \approx 23$
0.0044	0.133950	t=50
0.0010	0.133323	t=50
0.0000	0.133289	t=50

(a)

ε	$E(0)$	existence time
0.0010	0.133362	$t^* \approx 20.575$
0.0009	0.133355	$t^* \approx 24.575$
0.0008	0.133349	t=50
0.0005	0.133335	t=50
0.0000	0.133327	t=50

(b)

In Table 1 the qualitative properties of the solutions of (4) are given for different values of the initial energy $E(0)$ depending on the parameter ε. For $A = 0.495$ (Table 1a) the threshold between the global existence or finite blow up of the solutions is for some $\varepsilon_* \in (0.0044, 0.0045)$ and for $A = 0.499$ (Table 1b) the threshold is for $\varepsilon_* \in (0.0008, 0.0009)$. The corresponding energies for these values of ε_* are $E_*(0) \in (0.0133950, 0.133981)$ (Table 1a) and $E_*(0) \in (0.0133349, 0.133355)$ (Table 1b). Since $I(0) > 0$ this means that for energy $E_*(0) > d$ Theorem 1 is not true. Really, if Theorem 1 still holds for $E_*(0)$ then the solution should be globally defined which is not the case. The energy $E_*(0)$ is very closed to the critical energy $d \approx 0.1333\ldots$. The calculated values of $E_*(0)$ coincide up to the fourth and fifth digits respectively with the theoretical value $d \approx 0.1333\ldots$ found in Theorem 2. When the constant A tends to 0.5 and $A < 0.5$ the energy $E_*(0)$ will tend to d preserving the blow up properties of the solutions. Therefore the validity of Theorem 1 can not be extended for initial energy $E(0) > d$.

On Figure 1 the graphics of the solution of (4) for $A = 0.499$, $\varepsilon = 0.0008$ and $\varepsilon = 0.0009$ are illustrated. For $\varepsilon = 0.0008$ the solution stays bounded (Figure 1a) while for $\varepsilon = 0.0009$ (Figure 1b) the solution has typical blow up profile at time $t \approx 24.5$ very close to the blow up time $t^* \approx 24.575$. The numerical experiments in Table 2 confirm

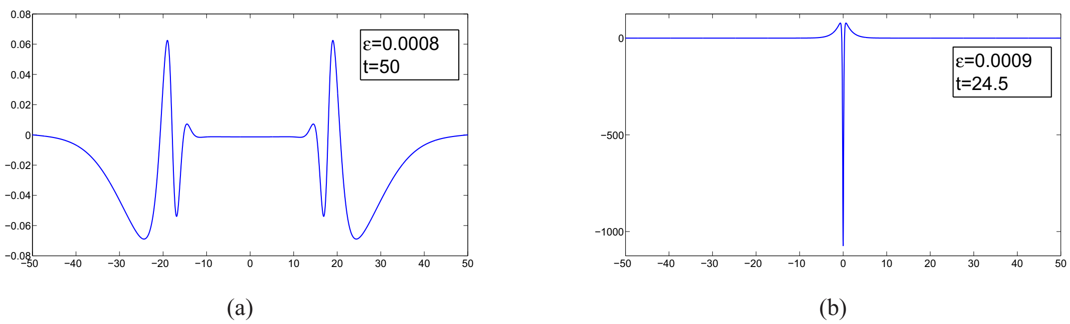


FIGURE 1. Profiles of the solution of (4) for $A = 0.499$ and $\varepsilon = 0.0008$ and $\varepsilon = 0.0009$

the previous results that the constant d is exact. For example, for $A = 0.505$ it follows that $I(0) < 0$. If Theorem 1 still holds for bigger initial energy $E(0) > d$ then the solution of (4) should blow up for finite time. However, for $\varepsilon = -0.0045, E(0) = 0.136199$ the solution is bounded for $t \leq 150$. In Figure 2, the graphics of the solution of (4) for $A = 0.505$ and $\varepsilon = -0.0045$ at different times $t = 0, 50, 100, 150$ are plotted. From the behavior of the solution it is clear that trough the evolution the solution stays bounded.

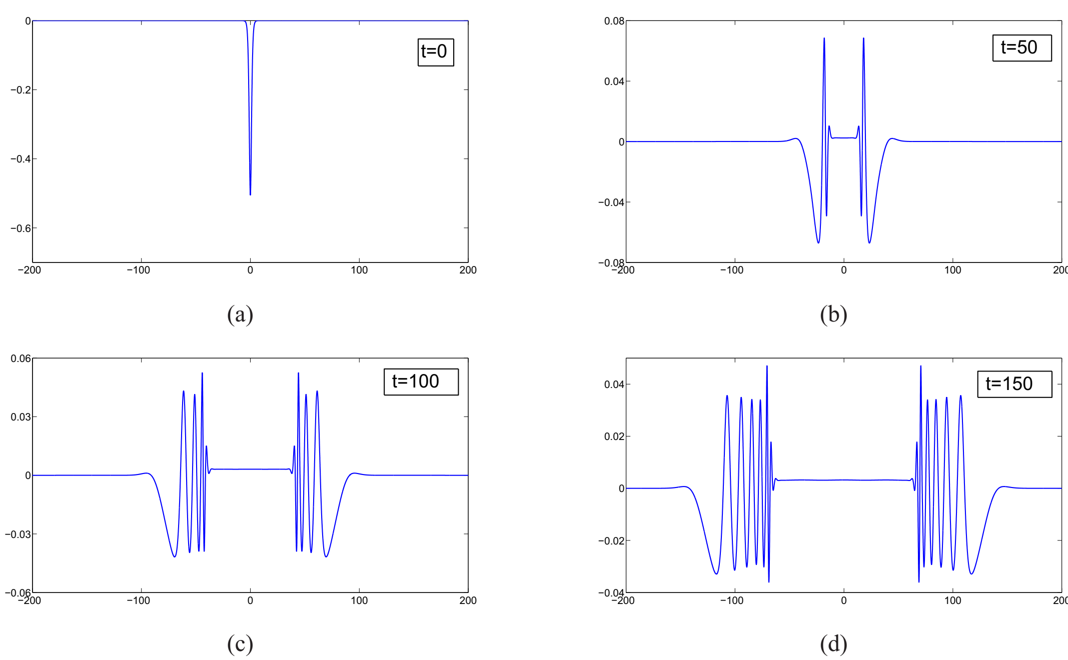


FIGURE 2. Evolution of the numerical solution of (4) for $A = 0.505$, $\varepsilon = -0.0045$ and $t \in [0, 150]$

The numerical experiments presented in Table 1 and Table 2 demonstrate that the constant d from Theorem 2 is the best possible one in 1D case. The careful analysis of the numerical results in Table 2 for $\varepsilon = 0.01$ and $\varepsilon = -0.01$

TABLE 2. Initial energy $E(0)$ as a function of ε and the existence time of the solutions of (4); $A = 0.505$, $I(0) \approx -0.008161 < 0$, $x \in [-200, 200]$, $0 \le t \le 150$, t^* - blow up time

ε	$E(0)$	existence time	ε	$E(0)$	existence time
0.0100	0.147663	$t^* \approx 9.975$	-0.0010	0.133432	$t^* \approx 12.925$
0.0000	0.133288	$t^* \approx 12.4$	-0.0044	0.136071	$t^* \approx 21.6$
-0.0100	0.147663	$t = 150$	-0.0045	0.136199	$t = 150$

with one and the same initial energy $E(0) = 0.147663$ shows a different behavior of the solutions. In the first case ($\varepsilon = 0.01$) the solution blows up for $t^* \approx 9.975$ and in the second case ($\varepsilon = -0.01$) the solution is bounded for $t \le 150$. The initial energy is greater than the critical one found in Theorem 2. Thus the only explanation of this phenomenon is the different signs of the initial velocities u_1. This means that some additional structure conditions on the data for large initial energies are necessary for global existence or finite blow up of the solution of (1).

Remark. A regular mesh with space step $h = 0.025$ and time step $t = 0.025$ is used for the numerical experiments demonstrated in Table 1 and Table 2.

APPLICATIONS

By means of the explicit value of d we will investigate the nonlinear stability of the solutions of BPE when $n = 1$, $p = 2$, $f(u) = \alpha u^2$, $\beta_1 \ge 0$, $\beta_2 > 0$, *i.e.*, for the equation

$$\begin{aligned} \beta_2 u_{tt} - u_{xx} - \beta_1 u_{ttxx} + u_{xxxx} &= \alpha(u^2)_{xx} \\ u(x,0) = u_0(x), \qquad u_t(x,0) &= u_1(x) \end{aligned} \tag{5}$$

It is wellknown (see [9],[5],[6]) that equation (5) has a solitary wave solution

$$w^c(x,t) = \frac{3}{2}\frac{(c^2-1)}{\alpha}\operatorname{sech}^2\left(\frac{1}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2 - \beta_2}}\left(\sqrt{\beta_2}x - ct\right)\right), \quad \text{for} \quad |c| < \min\{1, \sqrt{\frac{\beta_2}{\beta_1}}\},\ |c| > \max\{1, \sqrt{\frac{\beta_2}{\beta_1}}\}. \tag{6}$$

Simple calculations give us that the exact energy of the soliton w^c

$$E(0) = \frac{6}{5\alpha^2}\left|(1-c^2)(-5\frac{\beta_1}{\beta_2}c^4 + 4c^2 + 1)\right|\left|\frac{c^2-1}{(\frac{\beta_1}{\beta_2}c^2 - 1)}\right|^{\frac{1}{2}}, \tag{7}$$

which depends only on the ratio β_1/β_2, α and c.

We have the following stability result.

Theorem 3. *Suppose the initial energy (7) of the soliton w^c satisfies the inequality $E(0) < d$. Let u be a solution of (5) with initial data $u_0 \in \mathrm{H}$, $u_1 \in \mathrm{L}$. If the initial data u_0, u_1 are close to the soliton w^c,* i.e.*,*

$$\|u_0 - w^c(x,0)\|^2_{\mathrm{H}^1} < \varepsilon, \quad \|u_1 - w^c_t(x,0)\|^2_{\mathrm{L}^2} + \left\|(-\Delta)^{-1/2}(u_1 - w^c_t(x,0))\right\|^2_{\mathrm{L}^2} < \varepsilon$$

for sufficiently small positive ε then $E(u_0,u_1) < d$ and $I(u_0) > 0$ and consequently the solution u of (5) is defined and bounded for every $t \in [0,\infty)$.

Idea of the proof. Under the conditions of the Theorem 3 we will prove that $I(w^c) > 0$. If we suppose the opposite, *i.e.*, $I(w^c) < 0$, then from Theorem 1 (i) the soliton w^c should blow up for a finite time which contradicts the global existence of the soliton. If $I(w^c) = 0$ then from the definition of d we get the following impossible chain of inequalities

$$d = \inf_{\substack{u \in \mathrm{H}^1 \\ I(u)=0}} J(u) \le J(w^c) \le E(w^c) < d. \qquad \square$$

In the following three examples we illustrate Theorem 3 for the strong stability of a single solitary wave in 1D case for different values of the ratio β_1/β_2.
Example 1: $\beta_1/\beta_2 = 1$ In this case the solitary waves are defined for every $|c| \neq 1$. The initial energy of the soliton is

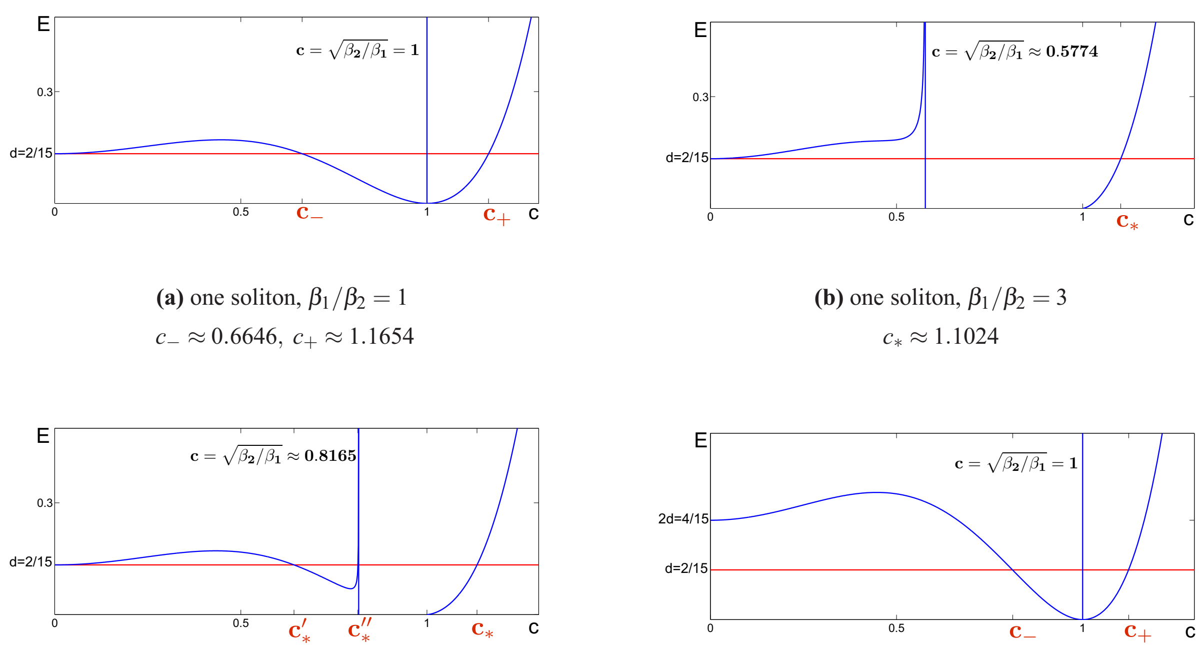
$$E(0) = \frac{6}{5\alpha^2}(1-c^2)(-5c^4+4c^2+1).$$

According to Figure 3a it follows that $E(0) < d$ for $c_- < |c| < c_+$, $|c| \neq 1$, $c_- \approx 0.6646$, $c_+ \approx 1.1654$. From Theorem 3 we have that the solution of (5) is defined and bounded in $[0,\infty)$ and consequently stable for initial data u_0, u_1 close to the solitary wave w^c with velocity $|c| \in (c_-, c_+)$.

What happens for $0 < |c| \leq c_-$ or $|c| \geq c_+$? In [21] the authors prove that for $0 < |c| < c_\delta < c_-$ the solution $u(x,t)$ of (5) with initial data close to the soliton blows up for a finite time. When $c_\delta \leq |c| \leq c_-$ and $|c| \geq c_+$ the behavior of the solution of BPE is an open problem.
Example 2: $\beta_1 > 0,\ \beta_2 > 0,\ \beta_1/\beta_2 > 1$ From formula (7) it follows that $E(0) < d$ when

$$\left|(1-c^2)(-5\frac{\beta_1}{\beta_2}c^4+4c^2+1)\right|\left|\frac{c^2-1}{(\frac{\beta_1}{\beta_2}c^2-1)}\right|^{\frac{1}{2}} < 1 \quad \text{for} \quad |c| \in (1, c_*), c_* = c_*(\beta_1, \beta_2).$$

On Figures 3b and 3c two particular cases of the ratio $\beta_1/\beta_2 = 3$ and $\beta_1/\beta_2 = 3/2$ are considered. In the first case ($\beta_1/\beta_2 = 3$) according to Theorem 3 the soliton is stable for $|c| \in (1, c_*)$, $c_* \approx 1.204$. In the second case ($\beta_1/\beta_2 = 3/2$) the stability of the soliton follows for velocities $|c| \in (c'_*, c''_*) \cup (1, c_*)$, where $c'_* \approx 0.6429$, $c''_* \approx 0.8145$ and $c_* \approx 1.1344$.



(a) one soliton, $\beta_1/\beta_2 = 1$
$c_- \approx 0.6646,\ c_+ \approx 1.1654$

(b) one soliton, $\beta_1/\beta_2 = 3$
$c_* \approx 1.1024$

(c) one soliton, $\beta_1/\beta_2 = 3/2$
$c'_* \approx 0.6429,\ c''_* \approx 0.8145,\ c_* \approx 1.1344$

(d) two solitons, $\beta_1/\beta_2 = 1$
$c_- \approx 8125,\ c_+ \approx 1.1238$

FIGURE 3. Exact energy E of the solution of (5) as a function of the velocity c

Example 3: Here we consider the collision of two solitons with opposite velocities and parameters satisfying $\beta_1/\beta_2 = 1$. The initial data are

$$u_0(x) = \left(w^c(x+x_0,t) + w^{-c}(x-x_0,t)\right)\Big|_{t=0}, \quad u_1(x) = \left(w^c_t(x+x_0,t) + w^{-c}_t(x-x_0,t)\right)\Big|_{t=0}. \tag{8}$$

We calculate the full energy

$$E(0) \approx 2\frac{6}{5\alpha^2}(1-c^2)(-5c^4+4c^2+1).$$

As in the previous examples we get $E(0) < d$ (see Figure 3d) for $|c| \in (c_-, c_+)$, $|c| \neq 1$, $c_- \approx 0.8125$, $c_+ \approx 1.1238$. According to Theorem 3 the stability of the two solitons with $|c| \in (c_-, c_+)$ follows, *i.e.*, the solution of the BPE with initial data u_0 and u_1 close to (8) is globally defined and bounded for every $t \in (0, \infty)$.

Theorem 3 and Examples 1-3 give us a new tool for the investigation of the stabilities of the solitons for different values of the dispersion parameters β_1 and β_2. Note that the ranges of the stability of the solitary waves can be found only numerically. The reason is that algebraic equations of high order can not be solvable in radicals.

This technique could be successfully applied to other nonlinearities $f(u) = \alpha|u|^p$, $p > 2$.

CONCLUSIONS

The constant d, explicitly found in Theorem 2, is the best possible one for the global existence or finite blow up of the solutions to BPE in 1D. The exact value of the this constant gives the boundaries of the application of the potential wells method and vacuum isolating of the solutions of BPE used in [20].

We demonstrate numerically the range of the stability of the solitons: the solution of BPE with data close to the initial profiles of the solitary waves are globally defined and bounded for $t \in [0, \infty)$.

Another conclusion from the numerical experiments is that the global solvability and finite blow up of the solutions depend on the "angle" between the initial data u_0 and u_1. More precisely in [16], [19] for $n \geq 1$, $\beta_1 = \beta_2 = 1$ there is a partial answer of this question, *i.e.*, the sign of

$$\beta_2\left((-\Delta)^{-1/2}u_1, (-\Delta)^{-1/2}u_0\right) + \beta_1(u_1, u_0) > 0$$

is important for the finite blow up of the solutions of (2).

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REFERENCES

1. J. V. Boussinesq (1871) *Comp. Rend. Hebd. des Seances de l'Acad des Sci.* **73**, 256–260.
2. J. V. Boussinesq (1872) *Journal de Mathématiques Pures et Appliquées* **17**, 55–108.
3. A. Chertock, C. I. Christov, and A. Kurganov (2011) *Computational Sci., & High performance Computing IV, NNFM* **113**, 267–281.
4. M. Christou and C. I. Christov, in *AMiTaNS'09*, edited by M.D. Todorov and C.I. Christov, AIP CP **1186**, American Institute of Physics, Melville, NY, 2009, pp. 217–224.
5. C. I. Cristov, in *Proc. ICFDS*, Oxford, 1996, pp. 343–349.
6. C. I. Christov (2001) *Wave motion* **34**, 161–174.
7. C. I. Christov, in *Differential Equations and Nonlinear Mechanics*, K. Vajravelu (Ed), 2001, pp. 63–77.
8. C. I. Christov, M. D. Todorov, and M. A. Christou, in *AMiTaNS'11*, edited by M.D. Todorov and C.I. Christov, American Institute of Physics, Melville, NY, 2011, accepted in this volume.
9. C I. Christov and M. Velarde (1994) *Intern. J Bifurcation Chaos* **4**, 1095–1112.
10. V. K. Kalantarov and O. A. Ladyzhenskaya (1977) *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya matematicheskogo Institute im. V. A. Steklova AN SSSR* **71**, 77–102. (in Russian)
11. H. A. Levine (1974) *TAMS* **192**, 1–21.
12. E. Lieb (1983) *Annals of Math.* **118**, 349–374.
13. Y. Lin and R. Xu (2008) *Physica D* **237**, 721–731.
14. Y. Liu (1995) *SIAM J. Math. Anal.* **26**, 1527–1546.
15. Y. Liu (2000) *J. Diff. Eq.* **164**, 223–2396.
16. N. Polat and A. Ertas (2009) *J. Math. Anal. Appl.* **349**, 10–20.
17. S. K. Turitzyn (1993) *Phys. Rev. E***47**, R13–R16.

18. S. K. Turitsyn (1993) *Phys. Rev. E* **47**, R769–R799.
19. S. Wang and G. Chen (2006) *Nonlinear Analysis* **64**, 159–173.
20. R. Xu and Y. Y. Liu (2009) *J. Math. Anal. Appl.* **359**, 739–751.
21. Y. Yang, C. Mu, and J. Deng (2008) *Nonlinear Analysis* **69**, 1599–1614.