DOI: 10.1007/978-3-642-18466-6_46

On the Numerical Simulation of Unsteady Solutions for the 2D Boussinesq Paradigm Equation

Christo I. Christov[1], Natalia Kolkovska[2], and Daniela Vasileva[2]

[1] Dept. of Mathematics, P.O. Box 41010, Lafayette, LA, 70504-1010, USA
christov@louisiana.edu
[2] Institute of Mathematics and Informatics, Bulgarian Acad. Sci., Acad. G. Bonchev str., bl.8, 1113 Sofia, Bulgaria
{natali,vasileva}@math.bas.bg

Abstract. For the solution of the 2D Boussinesq Paradigm Equation (BPE) an implicit, unconditionally stable difference scheme with second order truncation error in space and time is designed. Two different asymptotic boundary conditions are implemented: the trivial one, and a condition that matches the expected asymptotic behavior of the profile at infinity. The available in the literature solutions of BPE of type of stationary localized waves are used as initial conditions for different phase speeds and their evolution is investigated numerically. We find that, the solitary waves retain their identity for moderate times; for larger times they either transform into diverging propagating waves or blow-up.

1 Introduction

Boussinesq equation (BE) [1] is the first model for surface waves in shallow fluid layer that accounts for both nonlinearity and dispersion. The balance between the steepening effect of the nonlinearity and the flattening effect of the dispersion maintains the shape of the wave. In the 60s it was discovered that these permanent waves can behave in many instances as particles in 1D and they were called *solitons* by Zabusky and Kruskal [2]. It is of crucial importance to investigate also the 2D case, because of the different phenomenology and the practical importance. The accurate derivation of the Boussinesq system combined with an approximation, that reduces the full model to a single equation, leads to the Boussinesq Paradigm Equation (BPE) [3]:

$$u_{tt} = \Delta \left[u - F(u) + \beta_1 u_{tt} - \beta_2 \Delta u \right], \quad F(u) := \alpha u^2, \tag{1}$$

where u is the surface elevation, $\beta_1, \ \beta_2 > 0$ are two dispersion coefficients, and $\alpha > 0$ is an amplitude parameter. The main difference of (1) from BE is the presence of a term proportional to $\beta_1 \neq 0$ called "rotational inertia". Note that here we have changed the sign of the nonlinear term for the sake of the presentation.

I. Dimov, S. Dimova, and N. Kolkovska (Eds.): NMA 2010, LNCS 6046, pp. 386–394, 2011.
© Springer-Verlag Berlin Heidelberg 2011

It has been recently shown that the 2D BPE admits stationary translating localized solutions as well [4–7]. Even though no exact analytical formulas are available, those solutions can be accessed using either finite differences, perturbation technique, or Galerkin spectral method. However, virtually nothing is known about the dynamic properties of these solutions and their structural stability, i.e., what is their behavior when used as initial conditions for time-dependent computations of the BPE. The first results on this problem are reported in the pioneering work [8], but in order to investigate further the time evolution of the localized solutions, alternative techniques for Eq. (1) have to be developed.

2 Numerical Method for Solving BPE

In order to devise a numerical time-stepping procedure for Eq. (1), we set

$$v(x,y,t) := u - \beta_1 \Delta u. \tag{2a}$$

Upon substituting it in Eq. (1) we get the following equation for v

$$v_{tt} = \frac{\beta_2}{\beta_1}\Delta v + \frac{\beta_1 - \beta_2}{\beta_1^2}(u - v) - \Delta F(u). \tag{2b}$$

Now the system consists of an elliptic equation for u, Eq. (2a), and a hyperbolic equation for v: Eq. (2b). The system is inextricable coupled, because the function u is involved in the equation for v, and vice versa.

The following implicit time stepping can be designed for the system (2)

$$\begin{aligned}\frac{v_{ij}^{n+1} - 2v_{ij}^{n} + v_{ij}^{n-1}}{\tau^2} &= \frac{\beta_2}{2\beta_1}\Lambda\big[v_{ij}^{n+1} + v_{ij}^{n-1}\big] \\ &+ \frac{\beta_1 - \beta_2}{2\beta_1^2}[u_{ij}^{n+1} - v_{ij}^{n+1} + u_{ij}^{n-1} - v_{ij}^{n-1}] - \Lambda F(u_{ij}^{n}), \quad (3a)\\ u_{ij}^{n+1} - \beta_1 \Lambda u_{ij}^{n+1} &= v_{ij}^{n+1}, \quad i = 0, \ldots, N_x + 1, \quad j = 0, \ldots, N_y + 1. \quad (3b)\end{aligned}$$

Here τ is the time increment, and $\Lambda = \Lambda^{xx} + \Lambda^{yy}$ stands for the difference approximation of the Laplace operator Δ on a non-uniform grid, for example

$$\Lambda^{xx}\phi_{ij} = \frac{2\phi_{i-1j}}{h_{i-1}^x(h_i^x + h_{i-1}^x)} - \frac{2\phi_{ij}}{h_i^x h_{i-1}^x} + \frac{2\phi_{i+1j}}{h_i^x(h_i^x + h_{i-1}^x)} = \frac{\partial^2 \phi}{\partial x^2}\Big|_{ij} + O(|h_i^x - h_{i-1}^x|).$$

For a smooth distribution of the nonuniform grid (as the one considered here) one has

$$O(|h_i^x - h_{i-1}^x|) \approx \frac{\partial h^x}{\partial x} O(|h_{i-1}|^2) = O(|h_{i-1}|^2).$$

Respectively, the values of the sought functions at the $(n-1)$-st and n-th time stages are considered as known when computing the $(n+1)$-st stage. Thus, we have two *coupled* equations for the two unknown grid functions $u_{ij}^{n+1}, v_{ij}^{n+1}$ and use the following non-uniform grid in the $x-$direction

$$x_i = \sinh[\hat{h}_x(i - n_x)], \; x_{N_x+1-i} = -x_i, \; i = n_x + 1, \ldots, N_x + 1, \; x_{n_x} = 0,$$

where N_x is an odd number, $n_x = (N_x+1)/2$, $\hat{h}_x = D_x/N_x$, and D_x is selected in a manner to have large enough computational region. The grid in the $y-$direction is defined in the same way.

The unconditional stability of the scheme can be shown in a way, very similar to [9], where numerical experiments in the 1D case with the analogue of the scheme (3), confirm the findings in the literature (see, e.g. [10]) that the BPE solitons preserve their shape for all times and even after interaction.

In the simplest approximation, the boundary conditions can be set equal to zero, because of the localization of the wave profile. This forms the first set of b.c.'s used in the present work. However, the decay at infinity of the stationary propagating 2D Boussinesq solitons is second-order algebraic (see [4, 6]), which requires really large computational box in order that the solution in the main part of the region (far from the boundaries) is not adversely influenced. Thus, the second set of b.c.'s used in the present work are the asymptotic boundary conditions formulated in [7]

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} \approx -2u, \quad x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} \approx -2v, \qquad \sqrt{x^2+y^2} \gg 1. \tag{4}$$

We chose the following approximation for Eq. $(4)_1$ at the numerical infinities:

$$u^{n+1}_{i,N_y+1} = u^{n+1}_{i,N_y-1} + \frac{h^y_{N_y} + h^y_{N_y-1}}{y_{N_y}}\Big[-2u^{n+1}_{i,N_y} - \frac{x_i}{h^x_i + h^x_{i-1}}(u^{n+1}_{i+1,N_y} - u^{n+1}_{i-1,N_y})\Big],$$
$$u^{n+1}_{N_x+1,j} = u^{n+1}_{N_x-1,j} + \frac{h^x_{N_x} + h^x_{N_x-1}}{x_{N_x}}\Big[-2u^{n+1}_{N_x,j} - \frac{y_j}{h^y_j + h^y_{j-1}}(u^{n+1}_{N_x,j+1} - u^{n+1}_{N_x,j-1})\Big],$$

$i = 0,\dots,N_x,\ j = 0,\dots,N_y$. The implementation of Eq. $(4)_2$ is the same.

The initial conditions are created using the best-fit approximation provided in [6], and already used in [8]. The coupled system of equations (3) is solved by the Bi-Conjugate Gradient Stabilized Method with ILU preconditioner [11].

3 Numerical Experiments

Denote by $u^s(x,y;c)$ the best-fit approximation of the stationary translating (with speed c) localized solutions, obtained in [6]

$$\begin{aligned} u^s(x,y;c) &= f(x,y) + c^2\left[(1-\beta_1)g_a(x,y) + \beta_1 g_b(x,y)\right] \\ &\quad + c^2\left[(1-\beta_1)h_1(x,y) + \beta_1 h_2(x,y)\right]\cos\left[2\arctan(y/x)\right], \end{aligned}$$

where the formulas for the functions $f,\ g_a,\ g_b$ may be found in [6]. For $t=0$, the first initial condition is obvious: $u(x,y,0) = u^s(x,y;c)$, the second initial condition may be chosen as one of the following

$$\partial u/\partial t = -c\,\partial u^s/\partial y \quad \text{or} \quad u(x,y,-\tau) = u^s(x,y+c\tau;c), \tag{5}$$

and $(5)_1$ is approximated as $\dfrac{u^1_{ij} - u^{-1}_{ij}}{2\tau} = -c\,\dfrac{\partial u^s}{\partial y}(x_i,y_j)$.

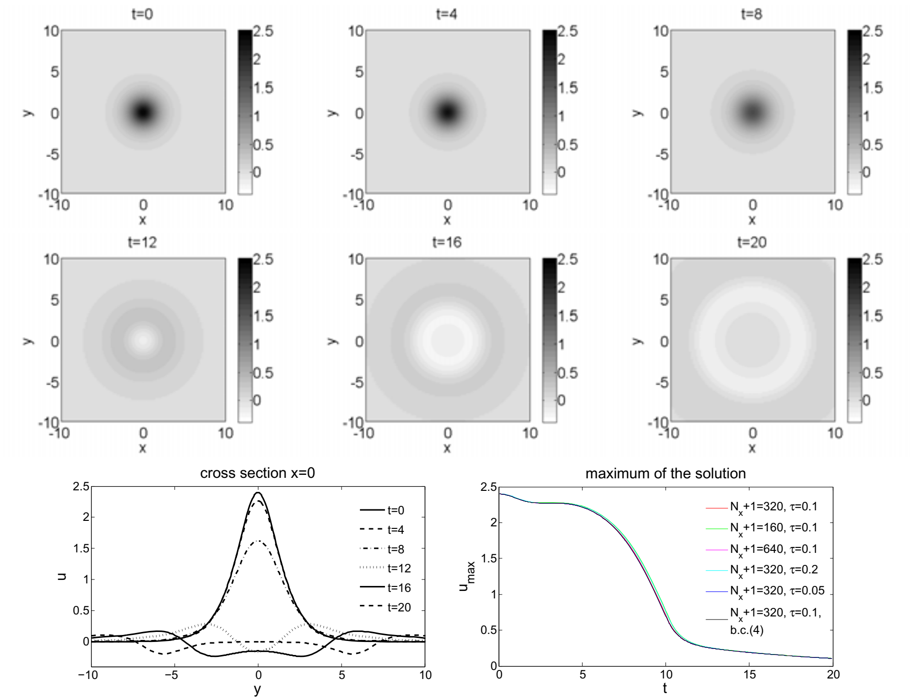

Fig. 1. Evolution of the solution for $c = 0$, the evolution of the cross-section at $x = 0$ and the values of the maximum

Table 1. The maximum of the solution, convergence in space and time, $c = 0$

		$t = 4$			$t = 8$			$t = 12$		
τ	N_x+1	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l
		with second IC according to $(5)_1$								
0.1	160	2.27122			1.64704			2.87575e-1		
0.1	320	2.26475	6.47e-3		1.60553	4.15e-2		2.80298e-1	7.28e-3	
0.1	640	2.26314	1.62e-3	2.0	1.59531	1.02e-2	2.0	2.78648e-1	1.65e-3	2.1
0.1	320	2.26475			1.60553			2.80298e-1		
0.05	320	2.26464	1.17e-4		1.60238	3.14e-3		2.79847e-1	4.51e-4	
0.025	320	2.26461	3.00e-5	2.0	1.60159	7.89e-4	2.0	2.79736e-1	1.11e-4	2.0
		with second IC according to $(5)_2$								
0.1	160	2.27016			1.62990			2.84523e-1		
0.1	320	2.26350	6.66e-3		1.58771	4.22e-2		2.77527e-1	6.996e-3	
0.1	640	2.26184	1.66e-3	2.0	1.57733	1.04e-2	2.0	2.75936e-1	1.591e-3	2.1
0.0125	320	2.26444			1.59915			2.79355e-1		
0.00625	320	2.26452	-7.80e-5		1.60022	-1.08e-3		2.79524e-1	-1.69e-4	
0.003125	320	2.26456	-4.00e-5	1.0	1.60077	-5.49e-4	1.0	2.79611e-1	-8.67e-5	1.0

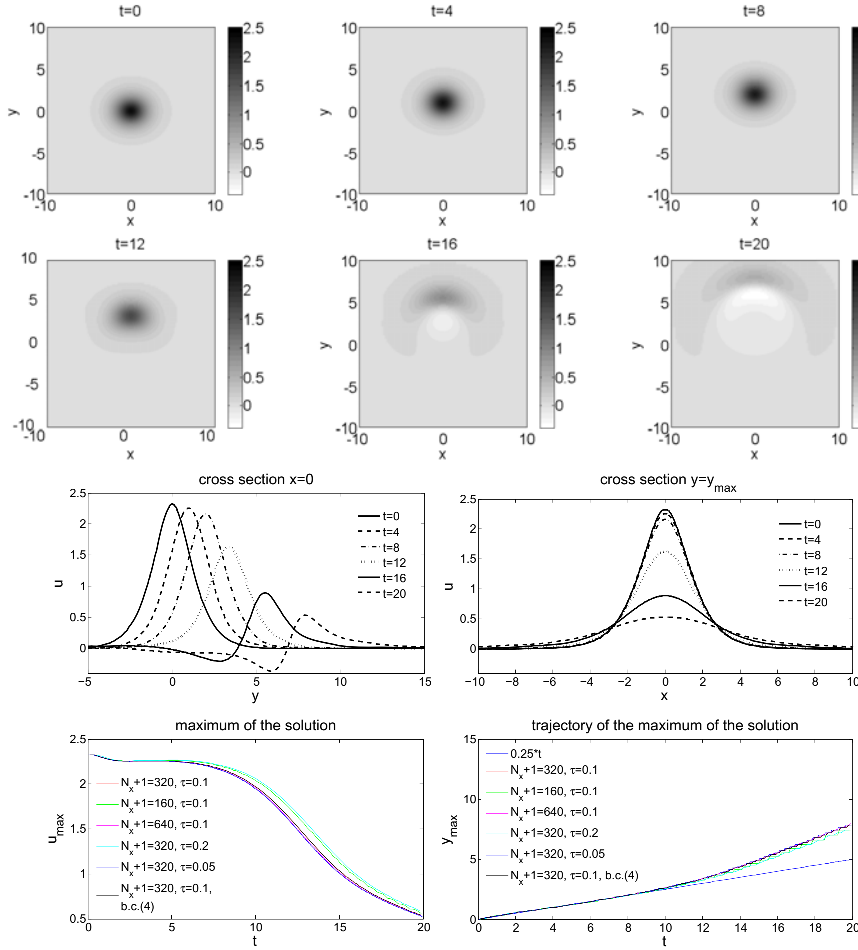

Fig. 2. Evolution of the solution for $c = 0.25$, evolution of the cross sections at $x = 0$ and $y = y_{\max}$, the maximum $u(0, y_{\max})$, and the trajectory of the maximum

The solutions for $\beta_1 = 3$, $\beta_2 = 1$, $\alpha = 1$ are computed on three different grids in the region $x, y \in [-50, 50]$ (with 161×161, 321×321 and 641×641 grid points), with at least three different time increments ($\tau = 0.2$, 0.1 and 0.05), and using either the trivial boundary conditions or the conditions (4).

Example 1. First, we present the results for the case $c = 0$, when the profile of the initial condition is a standing soliton. As it is seen in Fig. 1, the nonlinearity is not strong enough and after $t \geq 4$ the solution cannot keep the form, and eventually transforms into a propagating cylindrical wave, similar to the one generated on a water surface when an object is dropped into it (note, the sign of the solution is reversed in BPE (1)). The 'longitudinal' cross-section of the

Table 2. The maximum of the solution, convergence in space and time, $c = 0.25$

		$t=4$			$t=8$			$t=12$		
τ	N_x+1	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l	$u_{\max}$	$\Delta u_{\max}$	l
		with second IC according to $(5)_1$								
0.1	160	2.261156			2.191684			1.725273		
0.1	320	2.257642	3.51e-3		2.165738	2.59e-2		1.639348	8.59e-2	
0.1	640	2.256689	9.53e-4	1.9	2.158619	7.12e-3	1.9	1.619535	1.98e-2	2.1
0.2	320	2.268606			2.226354			1.848499		
0.1	320	2.257642	1.10e-2		2.165738	6.06e-2		1.639348	2.09e-1	
0.05	320	2.254871	2.77e-3	2.0	2.148196	1.75e-2	1.8	1.588800	5.05e-2	2.0
		with second IC according to $(5)_2$								
0.1	160	2.261550			2.189987			1.718885		
0.1	320	2.256804	4.75e-3		2.156155	3.38e-2		1.609205	1.10e-1	
0.1	640	2.255469	1.34e-3	1.8	2.147008	9.15e-3	1.9	1.584249	2.50e-2	2.1
0.2	320	2.264348			2.195763			1.734455		
0.1	320	2.256804	7.54e-3		2.156155	3.96e-2		1.609205	1.25e-1	
0.05	320	2.254958	1.85e-3	2.0	2.146491	9.66e-3	2.0	1.583778	2.54e-2	2.3

solution at $x = 0$ for a couple of moments of time and the values of the maximum of the solution as function of time are also shown in Fig. 1. The behaviour of the solution is the same on all grids, for all times steps, and does not depend on the type of the boundary conditions used (the trivial one or (4)).

For $t = 4, 8, 12$ the computed maximum of the solution $u_{\max}$, the difference $\Delta u_{\max} := u_{\max}^{\text{prev}} - u_{\max}$ (subscript 'prev' denotes the previous row in the table), and the rate of convergence $l = \log_2\left(|u_{\max}^{\text{prev}} - u_{\max}^{\text{prev,prev}}|/|u_{\max} - u_{\max}^{\text{prev}}|\right)$, are shown in Table 1. It is seen that when the second initial condition is taken according to $(5)_1$ the method is second order accurate in space and time. When the second initial condition is posed at $t = -\tau$ (i.e., $(5)_2$ is used), the method is only first order accurate in time, but this does not change significantly the behaviour of the solution, because the effect is localized near the initial moment of time.

Example 2. The case we discuss here is for $c = 0.25$. The results are presented in Fig. 2. The notation $y_{\max}$ is used for the y-coordinate of the maximum of the solution. For $t \leq 8$, the soliton not only moves with a speed, close to $c = 0.25$, but also behaves like a soliton, i.e., preserves its shape, albeit its maximum decreases slightly. For larger times, the solution transforms into a diverging propagating wave, but without a cylindrical symmetry: the fronts are deformed in the direction of propagation. As can be seen from Table 2 the method has second order numerical accuracy in space and time even when the second initial condition is posed at $t = -\tau$ (i.e., $(5)_2$ is used). This can be attributed to the fact that when $c = 0.25$ the solitary wave tends to preserve its shape, due to the inertia of motion, while for $c = 0$ the tendency towards diverging wave can onset in the very initial moment.

Example 3. In Fig. 3, results for $c = 0.3$ are presented which are second-order accurate in time, similarly to the case $c = 0.25$. For $t < 8$ the behavior of the

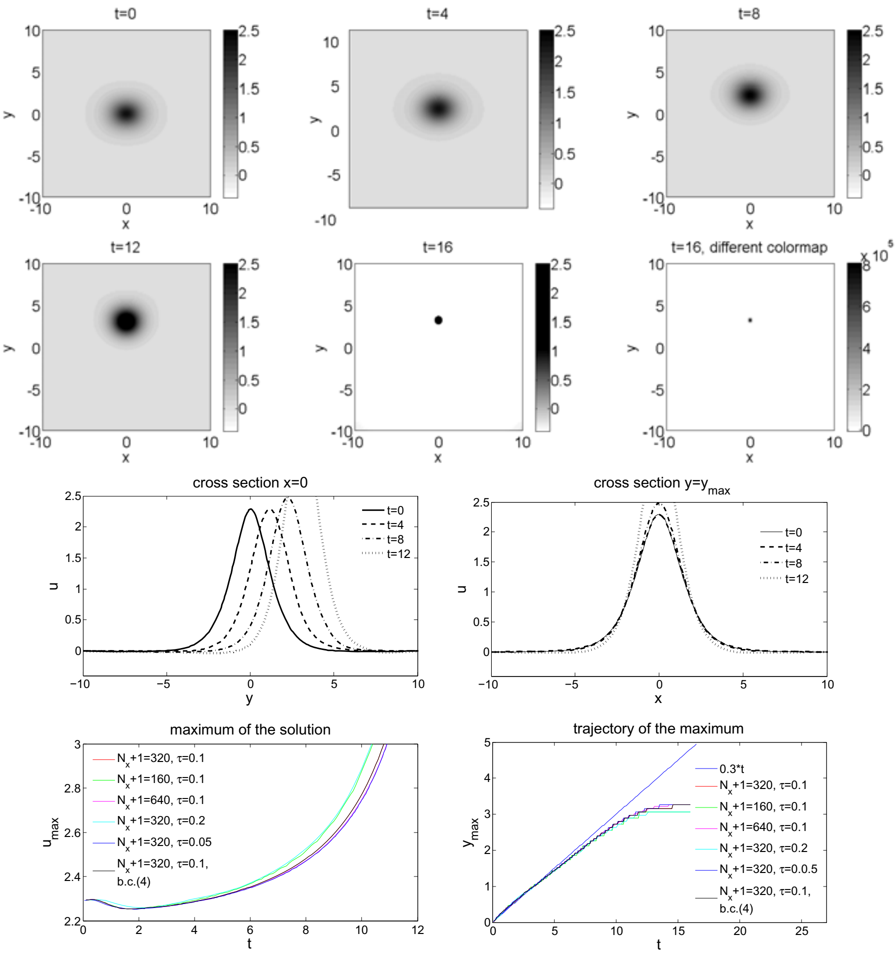

Fig. 3. Evolution of the solution for $c = 0.3$, the evolution of the cross sections at $x = 0$ and $y = y_{\max}$, the maximum $u(0, y_{\max})$, and the trajectory of the maximum

solution is similar to that in the previous example, but for larger times it turns to grow and blows-up for $t \approx 16$. The blow up is connected with the fact that the energy functional is not positive definite for BPE with quadratic nonlinearity (see [10] and the literature cited therein). A threshold value $c = 0.3$ was the last one for which a non-blowing-up evolution was found in [8] on the coarsest grid, while blow-up was encountered on the finest grid. Here we observe blow-up on all grids. This is probably due to the different numerical method used.

Example 4. Taking advantage of the efficiency of the algorithm presented here, we have taken the first sight into the interaction of two structures for different values of their phase speeds. The results are only preliminary, but they are important for answering the question of whether the stationary propagating

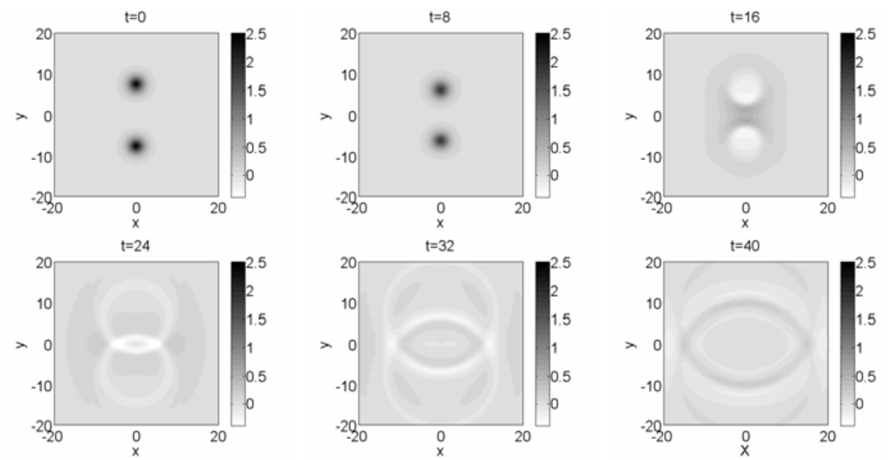

Fig. 4. Evolution of two interacting structures for $c = 0.15$

shapes are actually solitons if they are allowed to interact. In most of the cases with $c_1 = -c_2 \geq 0.2$ (and various initial distances between the structures), the solution blows up after the two structures clash. It is interesting that the threshold for the blow-up is lower than for the evolution of a single structure. We have been able to find non-blowing evolution for $c_1 = -c_2 = 0.15$, only when the initial distance between the centers of the structures is not very small, so the dispersion has some time to begin acting. The result is shown in Fig. 4, where the initial distance is 15. Indeed, the two structures have enough time to set on the track of dispersing waves (concentric diverging circles), and when the latter hit each other, a clear interference pattern onsets. The interaction is similar to the 1D case: they pass through each other. For the largest time $t = 40$ considered, the structures do not seem to have reemerged from the interaction because of their spread, but the centers of the 'rings' are well separated. In this sense the 2D structures under investigation can be termed 'aging coherent structures.' The detailed investigation of this issues requires a large set of numerical experiments, which goes beyond the frame of the present short note. What is important is that the developed here numerical tool is capable of solving the complex problem at hand.

4 Conclusion

In the present paper, a difference scheme for finding the time dependent localized solutions of the Boussinesq Paradigm Equation (BPE) in two spatial dimensions is devised. The grid is non-uniform and the truncation error is second order in space and time. To reduce the effects connected with the finite size of the computational domain, a special approximation of the asymptotic boundary conditions is used, in which the solution is matched to the expected asymptotic behavior at infinity.

In order to get insight into the possible quasi-particle (solitonic) behavior, results are obtained for the time evolution of supposedly stationary propagating waves for different phase speeds, whose profiles are available from the literature. We have found that for phase speeds $0 \neq c < 0.3$, the initially localized wave disperse in the form of ring-wave expanding to infinity. Respectively, for $c \geq 0.3$ the initial evolution resembles a stationary propagation, but after some period of time a blow-up of the solution takes place. This is in very good quantitative agreement with [8], where a similar (slightly higher threshold) is established for the appearance of the blow-up.

The fact that for $c \approx 0.3$, an time interval exists in which the solution is virtually preserving its shape whils steadily translating means that 2D solitons could be found in the class of the BPEs. This means that the nonlinearity is strong enough to balance the dispersion which is now much stronger than in the 1D case. In order to firmly establish this fact, our future plans are to consider also equation with different nonlinearity for which the blow-up is not possible.

References

1. Boussinesq, J.V.: Théorie des ondes et des remous qui se propagent le long d'un canal rectangulaire horizontal, en communiquant au liquide contenu dans ce canal des vitesses sensiblement pareilles de la surface au fond. Journal de Mathématiques Pures et Appliquées 17, 55–108 (1872)
2. Zabusky, N.J., Kruskal, M.D.: Interaction of 'solitons' in collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. 15, 240–243 (1965)
3. Christov, C.I.: An energy-consistent Galilean-invariant dispersive shallow-water model. Wave Motion 34, 161–174 (2001)
4. Christou, M.A., Christov, C.I.: Fourier-Galerkin method for 2D solitons of Boussinesq equation. Math. Comput. Simul. 74, 82–92 (2007)
5. Choudhury, J., Christov, C.I.: 2D solitary waves of Boussinesq equation. In: ISIS Int. Symp. Interdisc. Sci., Natchitoches 2004, APS Conf. Proc., vol. 755, pp. 85–90 (2005)
6. Christov, C.I., Choudhury, J.: Perturbation solution for the 2D shallow-water waves. Mech. Res. Commun. (submitted)
7. Christov, C.I.: Numerical implementation of the asymptotic boundary conditions for steadily propagating 2d solitons of Boussinesq type equations. Math. Comp. Simulat. (accepted)
8. Chertock, A., Christov, C.I., Kurganov, A.: Central-upwind schemes for the Boussinesq paradigm equation. In: Proc. 4th Russian-German Advanced Research Workshop on Computational Science and High Performance Computing (2010) (accepted)
9. Kolkovska, N.: Two Families of Finite Difference Schemes for Multidimensional Boussinesq Equation. In: AIP Conference Series (accepted)
10. Christov, C.I., Velarde, M.G.: Inelastic interaction of Boussinesq solitons. J. Bifurcation & Chaos 4, 1095–1112 (1994)
11. van der Vorst, H.: Iterative Krylov methods for large linear systems. Cambridge Monographs on Appl. and Comp. Math. 13 (2009)