The Space-Time Continuum as a Transversely Isotropic Material and the Meaning of the Temporal Coordinate

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Abstract. A transversely isotropic elastic continuum is considered in four dimensions, three of which are isotropic, and the properties of the material change only related to the fourth dimension. The model employs two dilational and three shear Lamé coefficients. The isotropic dilational coefficient is assumed to be much larger than the second dilational coefficient, and the three shear coefficients. This amounts to a material that is virtually incompressible in the three isotropic dimensions. The first and third shear coefficients are positive, while the second shear coefficient is assumed to be negative. As a result, in the equations of elastic equilibrium, the second derivatives of the displacement with respect to the fourth coordinate enter with negative sign. This makes the equations hyperbolic, with a fourth dimension opposing to the other three. The hyperbolic nature of the fourth dimension allows to be interpreted as time.

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INTRODUCTION

An important trait of the equations of electrodynamics and elasticity in the absence of dissipation is that when linearized they lead to hyperbolic (wave) equations. The most important trait of the wave equation is that it is hyperbolic, namely the second time derivative is “in opposition" to the second spatial derivatives, in the sense that they are on different sides of the equality sign in the equation. The similarity between time and space (both of them entering via second derivatives) is too conspicuous to be overlooked, but there remains an important difference: if all of the derivatives are lined together at the same side of the equality sign, the second time derivative has the sign opposite to the others. The idea to assume that time is just an imaginary pseudo-spatial variable is very intriguing. This is the idea behind the Einstein–Minkowski space-time and it has proven to lead to many quantitively adequate predictions, mostly stemming from the Lorentz Transformation (LT). The LT etched in stone the idea that time and space are interdependent, and this spawned what is known now as the Relativity Theory, the latter being considered to be the crown jewel of the Modern Physics of the 20th and 21st Centuries. Letting the product *it* denote the fourth spatial variable, allows one to have a plethora of invariant formulations in the *linear* case. However, since its very introduction the concept of space-time with one imaginary coordinate has struggled to attribute a clear physical meaning to the imaginary.

The relativity theory is riddled by internal logical paradoxes (see, e.g. [1, 2]), and most of the quantitative predictions show agreement with the experiments because of the fact that the LT emulates some of the convective terms in the convective time derivative that are not present in the linear governing equations of electrodynamics (see [3]). The present author suggested that space itself is a material manifold called the *metacontinuum*, and that what is perceived as particles and charges are localized deformations of the manifold. Assuming that the metacontinuum is an elastic liquid, allowed the author to derive Maxwell’s equations as rigorous corollaries from the governing equations of the elastic liquid under consideration [3]. The important new element of the unification of electrodynamics provided by the metacontinuum conjecture is that the laws of motional electrodynamics: Biot–Savart, Ampere–Oerstead, and Lorentz-force laws are *integral* parts of the model: they are related to the inertial effects in the metacontinuum. Assuming that the metacontinuum is a thin layer in the 4D space (a *hypershell*) brings into play the equation for the amplitude of the flexural deformations of the shell. This ‘master’ equation of the wave mechanics is nonlinear dispersive wave equation whose linear part is equivalent to the Shrödinger equation if written for the real (or imaginary) part of the wave function [4]. Considering the neutral particles as the solitons of the master wave equation explains the gravitation as the membrane tension, and the particle-wave dualism, as a natural property of the nonlinear localized flexural waves.

The success in unifying the electromagnetism, gravitation, and wave mechanics render a strong support for the notion that space is a material manifold, i.e., the laws of continuum mechanics govern all the processes in it. It seems

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important to investigate if the peculiar nature of the time variable may also be the result of the rheological properties of the metacontinuum. In the present short note we examine the possibility that the opposing nature of the time stems from the fact that we are faced with a transversely isotropic elastic continuum (see, e.g.[5], [6, §5.4], for definition), for which the three spatial direction are the isotropic ones, and along the fourth direction, the shear elastic modulus is different (actually with the opposite sign to the isotropic directions). In order to outline the main idea, we begin with an elastic body, not an elastic liquid. As shown in the cited authors works, the only difference is that in the case of a body, it is impossible to have stationary magnetic fields. If the new concept shows some promise, it will be straightforward afterwards to reformulate it for elastic liquids.

THE EQUATIONS OF ELASTIC EQUILIBRIUM

Consider a four-dimensional elastic continuum. We limit our scope to the case of a linear elasticity, when the referential description and the current configuration coincide. In order not to overload the presentation, we consider Cartesian coordinates. The first three coordinates x^1, x^2, x^3 are in the three dimensional subspace where the material is isotropic, while x^4 refers to the selected direction in which the material loses its isotropy. Let $\overline{\nabla} := (\nabla, \partial_{x^4}) \equiv (\nabla, \partial_4)$ stand the four dimensional gradient vector, while the notation $\nabla := (\partial_{x^1}, \partial_{x^2}, \partial_{x^3}) \equiv (\partial_1, \partial_2, \partial_3)$ is kept for the three dimensional gradient.

Regardless of the fact that we restricted ourselves to Cartesian coordinates, we will retain in the notations the contravariant and covariant nature of the different variables, which allows us to use the ubiquitous convention: if an index appears both as contravariant and covariant in a specific formula, summation is presumed.

Now, the different contravariant components of the four dimensional stress tensor $\mathfrak{S}$ as given by

$$\mathfrak{S} = \left(\begin{array}{c|c} \sigma^{ij} & \mathfrak{S}^{i4} \\ \hline \mathfrak{S}^{4j} & \mathfrak{S}^{44} \end{array} \right), \quad i,j = 1,2,3, \tag{1}$$

where $\boldsymbol{\sigma}$ is the three-dimensional Cauchy tensor.

The Cauchy balance for the four dimensional material $\overline{\nabla} \cdot \mathfrak{S} = 0$ can be expressed in the adopted notation as follows

$$\begin{aligned} \partial_{x^i} \sigma^{ij} + \partial_{x^4} \mathfrak{S}^{4j} &= 0, \quad j = 1,2,3, \\ \partial_{x^j} \mathfrak{S}^{4j} + \partial_{x^4} \mathfrak{S}^{44} &= 0. \end{aligned} \tag{2}$$

TRANSVERSELY ISOTROPIC MATERIAL IN FOUR DIMENSIONS

In this work, we consider a material whose response to shear stresses is transversely isotropic. Since the material manifold has four dimensions, three of which are isotropic (indistinguishable in the rheological sense), we introduce the pertinent notations which distinguish between the two groups of coordinates. Let $\boldsymbol{u} := (u^1, u^2, u^3)$ denote the three dimensional part of the displacement vector. Respectively, $\vec{u} := (\boldsymbol{u}, u^4) \equiv (u^1, u^2, u^3, u^4)$ stands for the full four dimensional displacement vector. Clearly, $\overline{\nabla}\vec{u}$ is a 4×4 matrix, while $\nabla \boldsymbol{u}$ is a 3×3 matrix. Then

$$\overline{\nabla}\vec{u} := \left(\begin{array}{c|c} \nabla \boldsymbol{u} & \partial_{x^4} \boldsymbol{u} \\ \hline \nabla u^4 & \partial_{x^4} u_4 \end{array} \right) \equiv \left(\begin{array}{c|c} \partial_{x^i} u^j & \partial_{x^4} u^j \\ \hline \partial_{x^i} u^4 & \partial_{x^4} u^4 \end{array} \right), \tag{3}$$

$$(\overline{\nabla}\vec{u})^T := \left(\begin{array}{c|c} (\nabla \boldsymbol{u})^T & \nabla u^4 \\ \hline \partial_{x^4} \boldsymbol{u}^T & \partial_{x^4} u^4 \end{array} \right), \tag{4}$$

$$\overline{\nabla} \cdot \vec{u} := \nabla \cdot \boldsymbol{u} + \partial_{x_4} u_4 \equiv \partial_{x^i} u^i + \partial_{x^4} u^4, \tag{5}$$

are the gradient, transposed gradient, and the divergence of the displacement vector $\vec{u}$.

The constitutive relationship with the proper symmetry reads

$$\boldsymbol{\sigma} = [(\lambda_1 \nabla \cdot \boldsymbol{u}) + \lambda_2 \frac{\partial u^4}{\partial x^4}]\mathbb{I} + \eta_1 \cdot (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T), \tag{6a}$$

$$\mathfrak{S}^{i4} = \mathfrak{S}^{4i} = \eta_2[\nabla u^4 + \partial_{x^4} \boldsymbol{u}], \tag{6b}$$

$$\mathfrak{S}^{44} = 2\eta_3 \partial_{x^4} u^4. \tag{6c}$$

One sees that any change of the coordinate axes in the isotropic dimensions do not change the 3D part of the constitutive relations. Similarly, the fourth dimension is "felt" through the derivative ∂_{x^4} and the scalar u^4 (the fourth component of the displacement vector), hence the respective terms are also invariant due to the scalar nature of this variable.

The above formulation shows that, similarly to the 3D case, one has merely five distinct coefficients in the constitutive relation if the material is presumed to be transversely isotropic. In this instance, the formulation presented here is much more concise than the engineering formulation (see, e.g. [6]). Apart from its actual relevance to the notion of time, our derivation is novel, and clarifies the way the transversally isotropic materials can be modeled.

Consequently, introducing Eqs. (6) in the Eqs. (2) we get

$$\eta_1 \nabla^2 \boldsymbol{u} + \eta_2 \frac{\partial^2 \boldsymbol{u}}{\partial (x^4)^2} + (\lambda_1 + 2\eta_1)\nabla(\nabla \cdot \boldsymbol{u}) + (\lambda_2 + \eta_2)\nabla \frac{\partial u^4}{\partial x^4} = 0, \tag{7a}$$

$$\eta_2(\nabla^2 u^4 + \partial_{x^4} \nabla \cdot \boldsymbol{u}) + 2\eta_3 \frac{\partial^2 u^4}{\partial (x^4)^2} = 0. \tag{7b}$$

In our previous works (see the literature cited in [3]), we have shown that the metacontinuum behaves as virtually incompressible medium, i.e., the dilational Lamé coefficient (connected to dilational elasticity modulus) is much larger than the shear Lamé coefficient (the latter being related to the speed of light). In the above notation this is expressed as $\lambda_1 \gg \eta_1$. We make the natural assumption that $\lambda_1 \gg \max\{\eta_1, \eta_2, \eta_3\}$. In such a case, a perturbation approach can be developed in the same vein as in the 3D case, and the Eqs (7) can be reduced in the first order of the small parameter $\varepsilon = \max\{\eta_1, \eta_2, \eta_3\}/\lambda$ to the following

$$-\eta_2 \frac{\partial^2 \boldsymbol{u}}{\partial (x^4)^2} = \eta_1 \nabla^2 \boldsymbol{u} + \nabla \phi, \tag{8a}$$

$$-2\eta_3 \frac{\partial^2 u^4}{\partial (x^4)^2} = \eta_2(\nabla^2 u^4 + \partial_{x^4} \nabla \cdot \boldsymbol{u}), \tag{8b}$$

$$\nabla(\nabla \cdot \boldsymbol{u}) = -\alpha \nabla \frac{\partial u^4}{\partial x^4},$$

where $\alpha = (\lambda_2 + \eta_2)/(\lambda_1 + 2\eta_1)$. The above equations contain the potential ϕ which has the meaning of pressure/tension that varies in the continuum to make the displacement field satisfy the last equation which is the continuity equation for the continuum. The latter cane be integrated once with respect to the spatial variables to obtain:

$$\nabla \cdot \boldsymbol{u} = -\alpha \frac{\partial u^4}{\partial x^4} + f(x^4), \tag{8c}$$

where the integration "constant" f is an arbitrary function of the fourth variable. In the absence of boundary condition it can be taken to be a true constant, which actually represents the density of the four dimensional continuum..

RELEVANCE TO THE NOTION OF TIME

Now, concerning the shear Lam'e coefficients, it has been established that η_1 is related to the electric permittivity (see [3] and the literature cited therein). Let us consider a transversely isotropic continuum with $\eta_2 < 0$ and introduce a new independent variable $t = \sqrt{|\eta_2|}x^4$. The Eq. (8a) can be rewritten as

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \eta_1 \nabla^2 \boldsymbol{u} + \nabla \phi, \tag{9a}$$

which is nothing else, but the momentum equations for an incompressible elastic continuum. Here, we can already obtain a self-contained model, if we assume that $\alpha \ll 1$. Then Eq. (8b) splits from the rest of the equations, while Eq. (8c) reduces to

$$\nabla \cdot \boldsymbol{u} = f(t), \tag{9b}$$

which is nothing else, but the continuity equation for incompressible linear elastic continuum for $f(t) = const$. The case $f \neq const$ describes an incompressible but not isochoric motion, i.e., the divergence of the displacement field is not influenced by the pressure, but can still vary with time (fourth dimension). In a sense, this can be interpreted as continuum that expands or shrinks due to external reasons, not included in the 3D governing equations. The rationale for taking $\alpha \ll 1$ is that the Maxwell equations can be derived in their well established form only if the 3D part of the metacontinuum is incompressible (see [3] for details) regardless to the presence of the fourth dimension. This can only happen if $\lambda_2 \ll \lambda_1$, otherwise, the derivative of the fourth component of the displacement will act in Eq. (8c) as a source term couple to the main equation. Thus, if a transversely isotropic elastic medium is considered as the model of the material manifold, then one can recover the linear elasticity in the three isotropic (spatial) dimension with the forth dimension (for which the shear Lamé coefficient is negative) playing the role of opposing (hyperbolic) dimension whose effect is perceived as temporal dimension.

THE CASE OF 4D INCOMPRESSIBILITY

When $\alpha \simeq O(1)$, Eq. (8c) gives a four dimensional incompressibility, which means that the three dimensional part of the motion is not necessarily isochoric. The interpretation is that the deformation $\partial_4 u^4$ in the fourth dimension can influence the motion in the three isotropic (spatial) dimensions acting as a "source" for the 3D motions. Now, introducing Eq. (8c) into Eq. (8b) gives

$$(-2\eta_3 + \alpha\eta_2)\frac{\partial^2 u^4}{\partial (x^4)^2} = \eta_2 \nabla^2 u^4 \tag{10}$$

We observe that the third shear coefficient η_3 is present in the left-hand side, while η_2 appears in the right-hand side. This requires choosing its sign: positive or negative. This issue calls for a judicious weighing of the different arguments: the different choices can yield qualitatively different models. In this first attempt, we begin with the model in which η_3 has the same sign as η_1, i.e., it is positive. Then, recalling the definition of the variable t, we get

$$\frac{\partial^2 u^4}{\partial t^2} = \beta \nabla^2 u^4, \quad \beta := \frac{\eta_2^2}{2\eta_3 - \alpha\eta_2} > 0, \tag{11}$$

which is a wave equation, i.e. it describes evolution with the fourth variable. Note that the speed of propagation of linear wave in the 3D momentum equations (9a) is $\sqrt{\eta_1}$, while the speed for the last equation is $\sqrt{\beta}$.

Now, if $\beta < 0$ the last equation is elliptic, which is a kind of non-causal model. The effect of the ellipticity of the model will result in some kind of "entanglement" or synchronicity of events separated by large intervals of time or large distances in space. At this stage there is not enough information in order to embark on a the quantitative development of such kind of a model.

CONCLUSIONS

The derivations presented here suffice to claim that the model of a transversely isotropic elastic continuum with one negative shear Lamé coefficient can explain the presence of a hyperbolic coordinate (the second derivatives with respect to which enter the equations with the opposite sign). This is a viable alternative to the Minkowski-space concept, in which the fourth coordinate is assumed to be imaginary in order to explain its "opposing" nature.

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